Theory and Examples

105. Vertical planes in cylindrical coordinates

a. Show that planes perpendicular to the x-axis have equations of the form $r = a \sec \theta$ in cylindrical coordinates.

b. Show that planes perpendicular to the y-axis have equations of the form $r = b \csc \theta$.

106. (*Continuation of Exercise 105.*) Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c$, $c \neq 0$.

107. Symmetry What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

108. Symmetry What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

15.8 Substitutions in Multiple Integrals

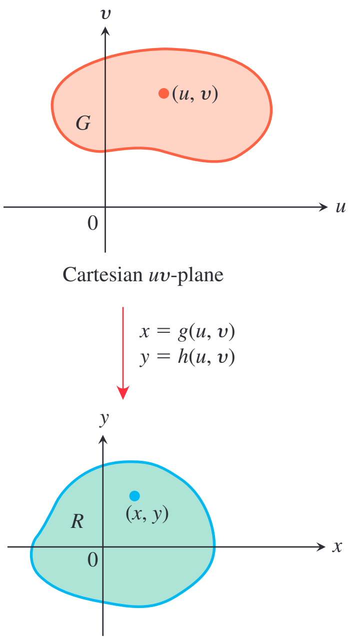

FIGURE 15.57 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy-plane into an integral over a region G in the uv-plane.

This section introduces the ideas involved in coordinate transformations to evaluate multiple integrals by substitution. The method replaces complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both. A thorough discussion of multivariable transformations and substitutions is best left to a more advanced course, but our introduction here shows how the substitutions just studied reflect the general idea derived for single integral calculus.

Substitutions in Double Integrals

The polar coordinate substitution of Section 15.4 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv-plane is transformed into the region R in the xy-plane by equations of the form

$$x = g(u, v), \qquad y = h(u, v),$$

as suggested in Figure 15.57. We assume the transformation is one-to-one on the interior of G. We call R the **image** of G under the transformation, and G the **preimage** of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G?

To gain some insight into the question, we look again at the single variable case. To be consistent with how we are using them now, we interchange the variables x and u used in the substitution method for single integrals in Chapter 5, so the equation is

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_a^b f(g(u))\, g'(u)\, du. \qquad x = g(u),\ dx = g'(u)du$$

To propose an analogue for substitution in a double integral $\iint_R f(x, y)\, dx\, dy$, we need a derivative factor like $g'(u)$ as a multiplier that transforms the area element $du\, dv$ in the region G to its corresponding area element $dx\, dy$ in the region R. We denote this factor by J. In continuing with our analogy, it is reasonable to assume that J is a function of both variables u and v, just as g' is a function of the single variable u. Moreover, J should register instantaneous change, so partial derivatives are going to be involved in its expression. Since four partial derivatives are associated with the transforming equations $x = g(u, v)$ and $y = h(u, v)$, it is also reasonable to assume that the factor $J(u, v)$ we seek includes them all. These features are captured in the following definition, which is constructed from the partial derivatives and is named after the German mathematician Carl Jacobi.

HISTORICAL BIOGRAPHY
Carl Gustav Jacob Jacobi (1804–1851)
www.bit.ly/2xZS8Wi

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v),\ y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \qquad (1)$$

The Jacobian can also be denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

to help us remember how the determinant in Equation (1) is constructed from the partial derivatives of x and y. The array of partial derivatives in Equation (1) behaves just like the derivative g' in the single variable situation. The Jacobian measures how much the transformation is expanding or contracting the area around the point (u, v). Effectively, the factor $|J|$ converts the area of the differential rectangle $du\, dv$ in G to match its corresponding differential area $dx\, dy$ in R. We note that, in general, the value of the scaling factor $|J|$ depends on the point (u, v) in G; that is, the scaling changes as the point (u, v) varies through the region G. Our examples to follow will show how it scales the differential area $du\, dv$ for specific transformations.

Differential Area Change Substituting $x = g(u, v),\ y = h(u, v)$

$$dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

Now we can answer our original question concerning the relationship of the integral of $f(x, y)$ over the region R to the integral of $f(g(u, v),\ h(u, v))$ over G.

THEOREM 3—Substitution for Double Integrals
Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v),\ y = h(u, v)$, which is assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v),\ h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \qquad (2)$$

The derivation of Equation (2) is intricate and properly belongs to a course in advanced calculus, so we do not include it here. We now present examples illustrating the substitution method defined by the equation.

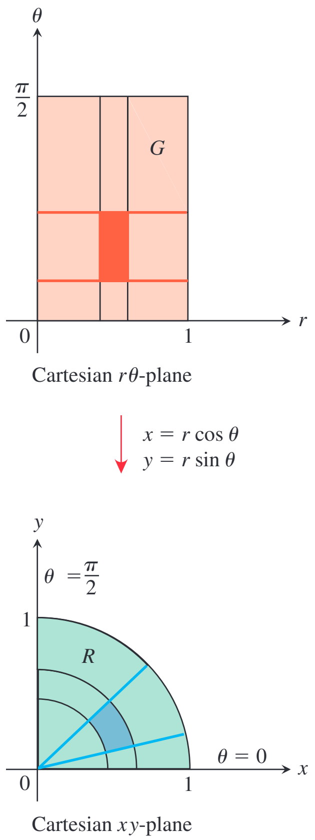

FIGURE 15.58 The equations $x = r\cos\theta,\ y = r\sin\theta$ transform G into R. The Jacobian factor r, calculated in Example 1, scales the differential rectangle $dr\, d\theta$ in G to match the differential area element $dx\, dy$ in R.

EXAMPLE 1 Find the Jacobian for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, and use Equation (2) to write the Cartesian integral $\iint_R f(x, y)\, dx\, dy$ as a polar integral.

Solution Figure 15.58 shows how the equations $x = r\cos\theta,\ y = r\sin\theta$ transform the rectangle $G\colon 0 \le r \le 1,\ 0 \le \theta \le \pi/2$, into the quarter of a circular disk R bounded by $x^2 + y^2 = 1$ in the first quadrant of the xy-plane.

For polar coordinates, we have r and θ in place of u and v. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Since we assume $r \geq 0$ when integrating in polar coordinates, $|J(r, \theta)| = |r| = r$ so that Equation (2) gives

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \tag{3}$$

This is the same formula we derived independently using a geometric argument for polar area in Section 15.4. ■

Here is an example of a substitution in which the image of a rectangle under the coordinate transformation is a trapezoid. Transformations like this one are called **linear transformations**, and their Jacobians are constant throughout G.

EXAMPLE 2 Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\,dx\,dy$$

by applying the transformation

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2} \tag{4}$$

and integrating over an appropriate region in the uv-plane.

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Figure 15.59).

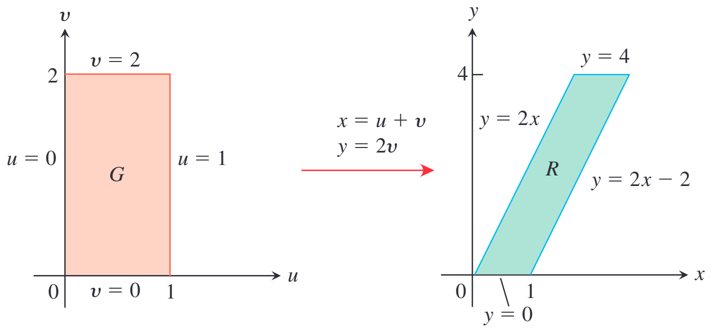

FIGURE 15.59 The equations $x = u + v$ and $y = 2v$ transform G into R. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G (Example 2).

To apply Equation (2), we need to find the corresponding uv-region G and the Jacobian of the transformation. To find them, we first solve Equations (4) for x and y in terms of u and v. From those equations it is easy to find algebraically that

$$x = u + v, \qquad y = 2v. \tag{5}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of R (Figure 15.59)

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

From Equations (5) the Jacobian of the transformation is

$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial}{\partial u}(u+v) & \dfrac{\partial}{\partial v}(u+v)\\[2mm] \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v)\end{vmatrix}=\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}=2.$$

We now have everything we need to apply Equation (2):

$$\int_0^4\int_{x=y/2}^{x=(y/2)+1}\frac{2x-y}{2}\,dx\,dy=\int_{v=0}^{v=2}\int_{u=0}^{u=1}u\,|J(u,v)|\,du\,dv$$

$$=\int_0^2\int_0^1(u)(2)\,du\,dv=\int_0^2\Big[u^2\Big]_{u=0}^{u=1}dv=\int_0^2 dv=2.$$ ■

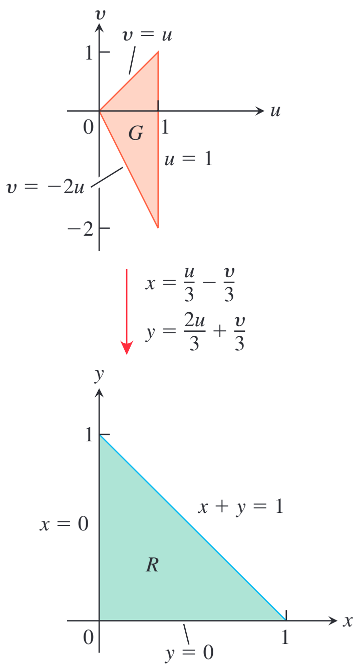

FIGURE 15.60 The equations $x=(u/3)-(v/3)$ and $y=(2u/3)+(v/3)$ transform G into R. Reversing the transformation by the equations $u=x+y$ and $v=y-2x$ transforms R into G (Example 3).

EXAMPLE 3 Evaluate

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Figure 15.60). The integrand suggests the transformation $u=x+y$ and $v=y-2x$. Routine algebra produces x and y as functions of u and v:

$$x=\frac{u}{3}-\frac{v}{3},\qquad y=\frac{2u}{3}+\frac{v}{3}. \tag{6}$$

From Equations (6), we can find the boundaries of the uv-region G (Figure 15.60).

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x+y=1$	$\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)=1$	$u=1$
$x=0$	$\frac{u}{3}-\frac{v}{3}=0$	$v=u$
$y=0$	$\frac{2u}{3}+\frac{v}{3}=0$	$v=-2u$

The Jacobian of the transformation in Equations (6) is

$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\dfrac{1}{3} & -\dfrac{1}{3}\\[2mm] \dfrac{2}{3} & \dfrac{1}{3}\end{vmatrix}=\frac{1}{3}.$$

Applying Equation (2), we evaluate the integral:

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dy\,dx=\int_{u=0}^{u=1}\int_{v=-2u}^{v=u}u^{1/2}v^2\,|J(u,v)|\,dv\,du$$

$$=\int_0^1\int_{-2u}^{u}u^{1/2}v^2\left(\frac{1}{3}\right)dv\,du=\frac{1}{3}\int_0^1 u^{1/2}\left[\frac{1}{3}v^3\right]_{v=-2u}^{v=u}du$$

$$=\frac{1}{9}\int_0^1 u^{1/2}(u^3+8u^3)\,du=\int_0^1 u^{7/2}\,du=\frac{2}{9}u^{9/2}\Big]_0^1=\frac{2}{9}.$$ ■

In the next example we illustrate a nonlinear transformation of coordinates resulting from simplifying the form of the integrand. Like the polar coordinates' transformation, nonlinear transformations can map a straight-line boundary of a region into a curved boundary (or vice versa with the inverse transformation). In general, nonlinear transformations are more complex to analyze than linear ones, and a complete treatment is left to a more advanced course.

EXAMPLE 4 Evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$$

Solution The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations gives $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$

with $u > 0$ and $v > 0$. Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

If G is the region of integration in the uv-plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G v e^u |J(u, v)|\, du\, dv = \iint_G v e^u \frac{2u}{v}\, du\, dv = \iint_G 2ue^u\, du\, dv.$$

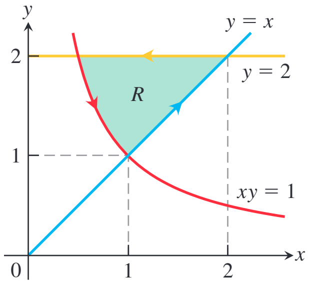

FIGURE 15.61 The region of integration R in Example 4.

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration R of the original integral in the xy-plane is shown in Figure 15.61. From the substitution equations $u = \sqrt{xy}$ and $v = \sqrt{y/x}$, we see that the image of the left-hand boundary $xy = 1$ for R is the vertical line segment $u = 1$, $2 \geq v \geq 1$, in G (see Figure 15.62). Likewise, the right-hand boundary $y = x$ of R maps to the horizontal line segment $v = 1$, $1 \leq u \leq 2$, in G. Finally, the horizontal top boundary $y = 2$ of R maps to $uv = 2$, $1 \leq v \leq 2$, in G. As we move counterclockwise around the boundary of the region R, we also move counterclockwise around the boundary of G, as shown in Figure 15.62. Knowing the region of integration G in the uv-plane, we can now write equivalent iterated integrals:

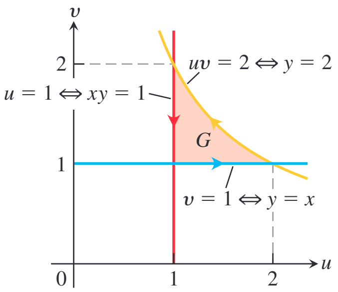

FIGURE 15.62 The boundaries of the region G correspond to those of region R in Figure 15.61. Notice that as we move counterclockwise around the region R, we move counterclockwise around the region G as well. The inverse transformation equations $u = \sqrt{xy}$, $v = \sqrt{y/x}$ produce the region G from the region R.

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du. \qquad \text{Note the order of integration.}$$

We now evaluate the transformed integral on the right-hand side:

$$\begin{aligned} \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du &= 2\int_1^2 \Big[vue^u \Big]_{v=1}^{v=2/u} du \\ &= 2\int_1^2 (2e^u - ue^u)\, du \\ &= 2\int_1^2 (2 - u)e^u\, du \\ &= 2\Big[(2 - u)e^u + e^u\Big]_{u=1}^{u=2} \qquad \text{Integrate by parts.} \\ &= 2(e^2 - (e + e)) = 2e(e - 2). \end{aligned}$$

■

Substitutions in Triple Integrals

The cylindrical and spherical coordinate substitutions in Section 15.7 are special cases of a substitution method that pictures changes of variables in triple integrals as transformations of solid regions. The method is like the method for double integrals given by Equation (2) except that now we work in three dimensions instead of two.

Suppose that a solid region G in uvw-space is transformed one-to-one into the solid region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

as suggested in Figure 15.63. Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w),\ h(u, v, w),\ k(u, v, w)) = H(u, v, w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G by the equation

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(u, v, w)\,|J(u, v, w)|\, du\, dv\, dw. \qquad (7)$$

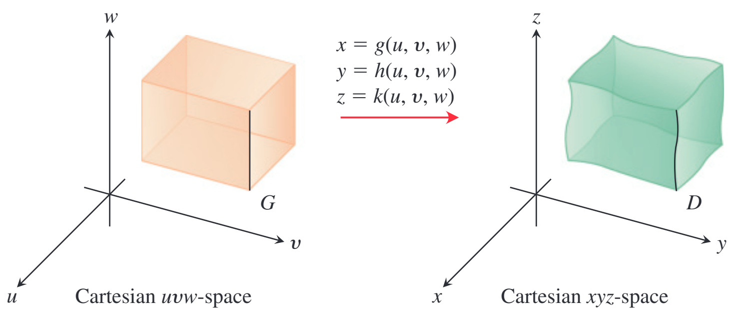

FIGURE 15.63 The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region D in Cartesian xyz-space into an integral over a region G in Cartesian uvw-space using Equation (7).

Determinants
2×2 and 3×3 determinants are evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The factor $J(u, v, w)$, whose absolute value appears in this equation, is the **Jacobian determinant**

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates. As in the two-dimensional case, the derivation of the change-of-variable formula in Equation (7) is omitted.

For cylindrical coordinates, r, θ, and z take the place of u, v, and w. The transformation from Cartesian $r\theta z$-space to Cartesian xyz-space is given by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

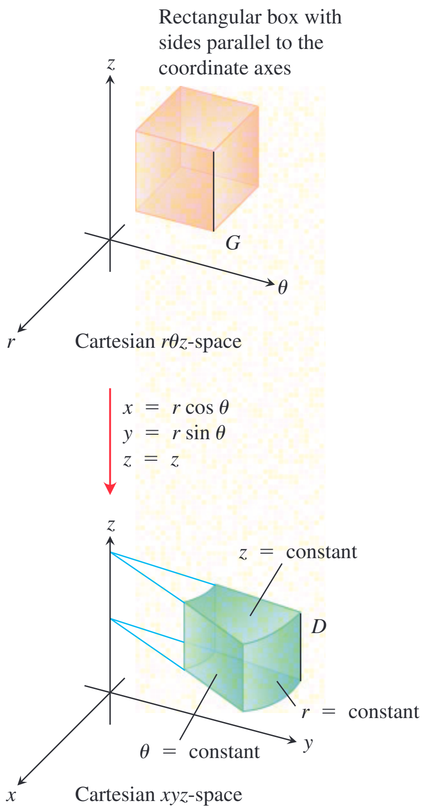

FIGURE 15.64 The equations $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ transform the rectangular box G into a cylindrical wedge D.

(Figure 15.64). The Jacobian of the transformation is

$$J(r, \theta, z) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(r, \theta, z)\,|r|\, dr\, d\theta\, dz.$$

We can drop the absolute value signs because $r \geq 0$.

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is given by

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$$

(Figure 15.65). The Jacobian of the transformation (see Exercise 23) is

$$J(\rho, \phi, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2\sin\phi.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(\rho, \phi, \theta)\,|\rho^2\sin\phi|\, d\rho\, d\phi\, d\theta.$$

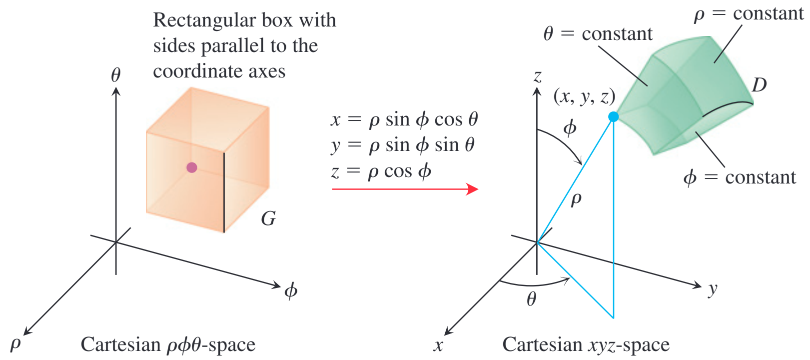

FIGURE 15.65 The equations $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ transform the rectangular box G into the spherical wedge D.

We can drop the absolute value signs because $\sin\phi$ is never negative for $0 \leq \phi \leq \pi$. Note that this is the same result we obtained in Section 15.7.

Here is an example of another substitution. Although we could evaluate the integral in this example directly, we have chosen it to illustrate the substitution method in a simple (and fairly intuitive) setting.

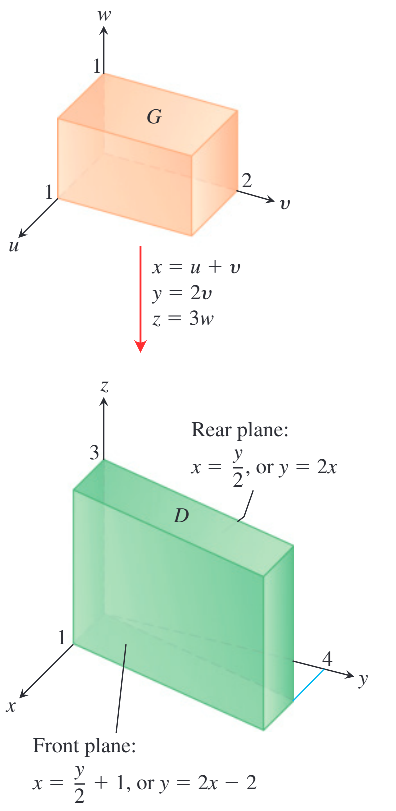

FIGURE 15.66 The equations $x = u + v$, $y = 2v$, and $z = 3w$ transform G into D. Reversing the transformation by the equations $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ transforms D into G (Example 5).

EXAMPLE 5 Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

by applying the transformation

$$u = (2x - y)/2, \qquad v = y/2, \qquad w = z/3 \tag{8}$$

and integrating over an appropriate region in uvw-space.

Solution We sketch the solid region D of integration in xyz-space and identify its boundaries (Figure 15.66). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding uvw-region G and the Jacobian of the transformation. To find them, we first solve Equations (8) for x, y, and z in terms of u, v, and w. Routine algebra gives

$$x = u + v, \qquad y = 2v, \qquad z = 3w. \tag{9}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D.

xyz-equations for the boundary of D	Corresponding uvw-equations for the boundary of G	Simplified uvw-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned}
&\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w) \left| J(u, v, w) \right| du\, dv\, dw \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\, du\, dv\, dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_{u=0}^{u=1} dv\, dw \\
&= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\, dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_{v=0}^{v=2} dw = 6 \int_0^1 (1 + 2w)\, dw \\
&= 6 \left[w + w^2 \right]_0^1 = 6(2) = 12.
\end{aligned}$$

■

EXERCISES 15.8

Jacobians and Transformed Regions in the Plane

1. **a.** Solve the system

$$u = x - y, \qquad v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

2. **a.** Solve the system

$$u = x + 2y, \qquad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy-plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv-plane.

3. **a.** Solve the system

$$u = 3x + 2y, \qquad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

4. **a.** Solve the system

$$u = 2x - 3y, \qquad v = -x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy-plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv-plane.

Substitutions in Double Integrals

5. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy$$

from Example 1 directly by integration with respect to x and y to confirm that its value is 2.

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y)\, dx\, dy.$$

9. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

10. **a.** Find the Jacobian of the transformation $x = u$, $y = uv$ and sketch the region $G: 1 \le u \le 2$, $1 \le uv \le 2$, in the uv-plane.

b. Then use Equation (2) to transform the integral

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$

into an integral over G, and evaluate both integrals.

11. **Polar moment of inertia of an elliptical plate** A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$, in the xy-plane. Find the first moment of the plate about the origin. (*Hint:* Use the transformation $x = ar\cos\theta$, $y = br\sin\theta$.)

12. **The area of an ellipse** The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk $G: u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

13. Use the transformation in Exercise 2 to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)\, e^{(y-x)}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y)\, e^{(2x-y)^2}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

15. Use the transformation $x = u/v$, $y = uv$ to evaluate the integral sum

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dx\, dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dx\, dy.$$

16. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\, dy\, dx.$$

(*Hint:* Show that the image of the triangular region G with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Substitutions in Triple Integrals

17. Evaluate the integral in Example 5 by integrating with respect to x, y, and z.

18. Volume of a solid ellipsoid Find the volume of the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw-space.)

19. Evaluate

$$\iiint_D |xyz|\, dx\, dy\, dz$$

over the solid ellipsoid D,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then integrate over an appropriate region in uvw-space.)

20. Let D be the solid region in xyz-space defined by the inequalities

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2 y + 3xyz)\, dx\, dy\, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

Theory and Examples

21. Find the Jacobian $\partial(x, y)/\partial(u, v)$ of the transformation

a. $x = u \cos v, \quad y = u \sin v$

b. $x = u \sin v, \quad y = u \cos v.$

22. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation

a. $x = u \cos v, \quad y = u \sin v, \quad z = w$

b. $x = 2u - 1, \quad y = 3v - 4, \quad z = (1/2)(w - 4).$

23. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is $\rho^2 \sin \phi$.

24. Substitutions in single integrals How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

25. Centroid of a solid semi-ellipsoid Assuming the result that the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semi-ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1$, $z \geq 0$, lies on the z-axis three-eighths of the way from the base toward the top. (You can do this without evaluating any of the integrals.)

26. Cylindrical shells In Section 6.2, we learned how to find the volume of a solid of revolution using the shell method. Specifically, if the region between the curve $y = f(x)$ and the x-axis from a to b $(0 < a < b)$ is revolved about the y-axis, the volume of the resulting solid is $\int_a^b 2\pi x f(x)\, dx$. Prove that finding volumes by using triple integrals gives the same result. (*Hint:* Use cylindrical coordinates with the roles of y and z changed.)

27. Inverse transform The equations $x = g(u, v)$, $y = h(u, v)$ in Figure 15.57 transform the region G in the uv-plane into the region R in the xy-plane. Since the substitution transformation is one-to-one with continuous first partial derivatives, it has an inverse transformation, and there are equations $u = \alpha(x, y)$, $v = \beta(x, y)$ with continuous first partial derivatives transforming R back into G. Moreover, the Jacobian determinants of the transformations are related reciprocally by

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1}. \tag{10}$$

Equation (10) is proved in advanced calculus. Use it to find the area of the region R in the first quadrant of the xy-plane bounded by the lines $y = 2x$, $2y = x$, and the curves $xy = 2$, $2xy = 1$ for $u = xy$ and $v = y/x$.

28. (*Continuation of Exercise 27.*) For the region R described in Exercise 27, evaluate the integral $\iint_R y^2\, dA$.

CHAPTER 15 Questions to Guide Your Review

1. Define the double integral of a function of two variables over a bounded region in the coordinate plane.

2. How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? Give examples.

3. How are double integrals used to calculate areas and average values. Give examples.

4. How can you change a double integral in rectangular coordinates into a double integral in polar coordinates? Why might it be worthwhile to do so? Give an example.

5. Define the triple integral of a function $f(x, y, z)$ over a bounded solid region in space.

6. How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? Give an example.

7. How are double and triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, and centers of mass? Give examples.

8. How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates?

9. How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? Give examples.

10. How are substitutions in double integrals pictured as transformations of regions in the plane? Give a sample calculation.

11. How are substitutions in triple integrals pictured as transformations of solid regions? Give a sample calculation.