b. Use Equation (2) and the result in part (a) to find the moment of inertia of the solid about the line $x = 0$, $y = 2b$.

38. If $a = b = 6$ and $c = 4$, the moment of inertia of the solid wedge in Exercise 22 about the x-axis is $I_x = 208$. Find the moment of inertia of the wedge about the line $y = 4$, $z = -4/3$ (the edge of the wedge's narrow end).

Joint Probability Density Functions

For Exercises 39–42, verify that f gives a joint probability density function. Then find the expected values μ_X and μ_Y.

39. $f(x, y) = \begin{cases} x + y, & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$

40. $f(x, y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$

41. $f(x, y) = \begin{cases} 6x^2y, & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$

42. $f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$

43. Suppose that f is a uniform joint probability density function on $0 \le x < 2$, $0 \le y < 3$. What is the formula for f? What is the probability that $X < Y$?

44. The following formula defines a joint probability density function. What is the value of C? What are the expected values μ_X and μ_Y?

$$f(x, y) = \begin{cases} Cxy, & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

15.7 Triple Integrals in Cylindrical and Spherical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane discussed in Section 15.4.

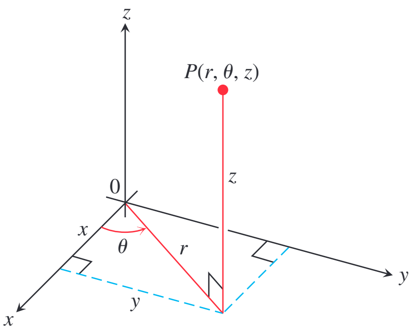


FIGURE 15.46 The cylindrical coordinates of a point in space are r, θ, and z.

Integration in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space coordinate triples of the form (r, θ, z), as shown in Figure 15.46. Here we require $r \ge 0$.

> **DEFINITION** **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which
>
> **1.** r and θ are polar coordinates for the vertical projection of P on the xy-plane, with $r \ge 0$, and
>
> **2.** z is the rectangular vertical coordinate.

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

> **Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates**
>
> $$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
> $$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

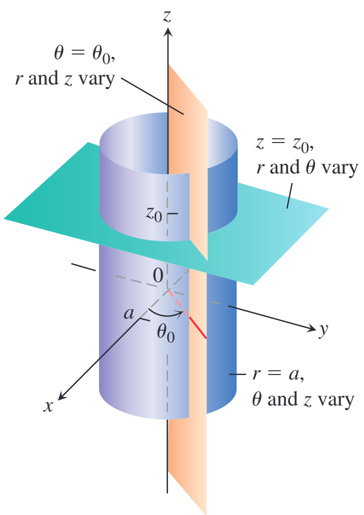


FIGURE 15.47 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Figure 15.47). The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$$
\begin{aligned}
r &= 4 && \text{Cylinder, radius 4, axis the } z\text{-axis} \\
\theta &= \frac{\pi}{3} && \text{Half-plane containing the } z\text{-axis} \\
z &= 2. && \text{Plane perpendicular to the } z\text{-axis}
\end{aligned}
$$

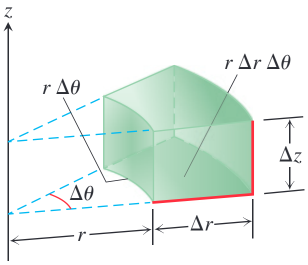


FIGURE 15.48 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = r\,\Delta z\,\Delta r\,\Delta\theta$.

When computing triple integrals over a solid region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In the kth cylindrical wedge, r, θ, and z change by Δr_k, $\Delta\theta_k$, and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the **norm** of the partition. We express the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz_k (Figure 15.48).

For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculated in polar coordinates that $\Delta A_k = r_k\,\Delta r_k\,\Delta\theta_k$. So $\Delta V_k = \Delta z_k\,r_k\,\Delta r_k\,\Delta\theta_k = r_k\,\Delta z_k\,\Delta r_k\,\Delta\theta_k$, and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\,r_k\,\Delta z_k\,\Delta r_k\,\Delta\theta_k.$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

Volume Differential in Cylindrical Coordinates

$$dV = r\,dz\,dr\,d\theta$$

$$\boxed{\lim_{n\to\infty} S_n = \iiint_D f\,dV = \iiint_D f\,r\,dz\,dr\,d\theta.}$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example. Although the definition of cylindrical coordinates makes sense without any restrictions on θ, in most situations when integrating, we will need to restrict θ to an interval of length 2π. So we impose the requirement that $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.

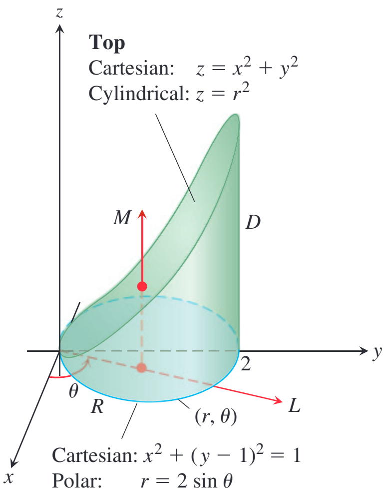


FIGURE 15.49 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the solid region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y-1)^2 = 1$. Its polar coordinate equation is

$$
\begin{aligned}
x^2 + (y-1)^2 &= 1 \\
x^2 + y^2 - 2y + 1 &= 1 \\
r^2 - 2r\sin\theta &= 0 \\
r &= 2\sin\theta.
\end{aligned}
$$

The region is sketched in Figure 15.49.

We find the limits of integration, starting with the z-limits. A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2\sin\theta$.

Finally, we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z)\, r\, dz\, dr\, d\theta.$$

Example 1 illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized as follows.

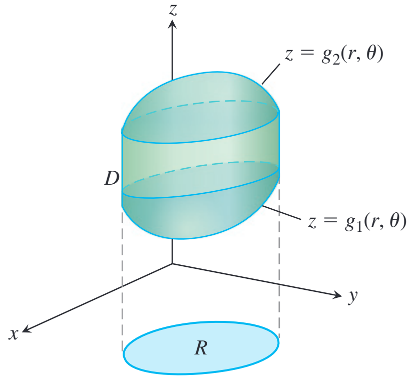
How to Integrate in Cylindrical Coordinates

To evaluate

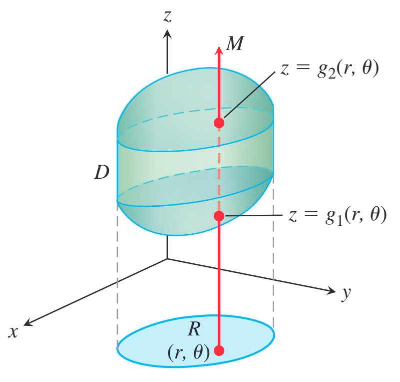
$$\iiint_D f(r, \theta, z)\, dV$$

over a solid region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

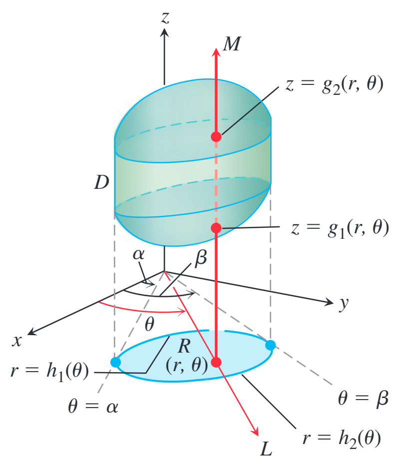
1. *Sketch*. Sketch the solid region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2. *Find the z-limits of integration.* Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



4. *Find the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, r\, dz\, dr\, d\theta.$$

EXAMPLE 2 Find the centroid $(\delta = 1)$ of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

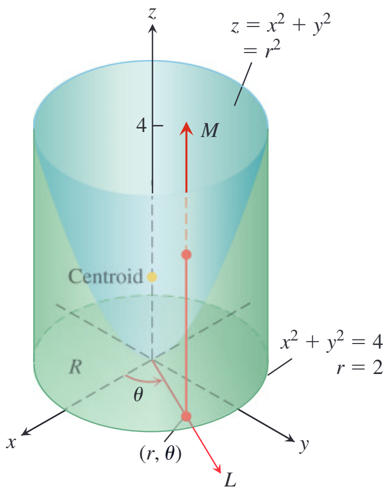


FIGURE 15.50 Example 2 shows how to find the centroid of this solid.

Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 15.50). Its base R is the disk $0 \le r \le 2$ in the xy-plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r-limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ-limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2}\right]_{z=0}^{z=r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{r^5}{2}\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^6}{12}\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \frac{16}{3}\, d\theta = \frac{32\pi}{3}.$$

The value of M is

$$M = \int_0^{2\pi} \int_0^2 \int_0^{r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[z\right]_{z=0}^{z=r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4}\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi.$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies on the z-axis, outside the solid. ■

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles and one distance, as shown in Figure 15.51. The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin and is never negative. The second coordinate, ϕ, is the angle $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval $[0, \pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

ϕ is the Greek letter phi, pronounced "fee."

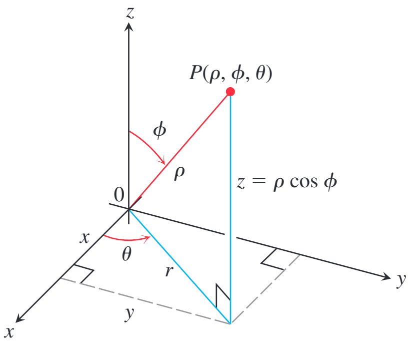


FIGURE 15.51 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin $(\rho \geq 0)$.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates.

On maps of Earth, θ is related to the longitude of a point on the planet and ϕ to its latitude, while ρ is related to elevation above Earth's surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin (Figure 15.52). The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \pi/2$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward. The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

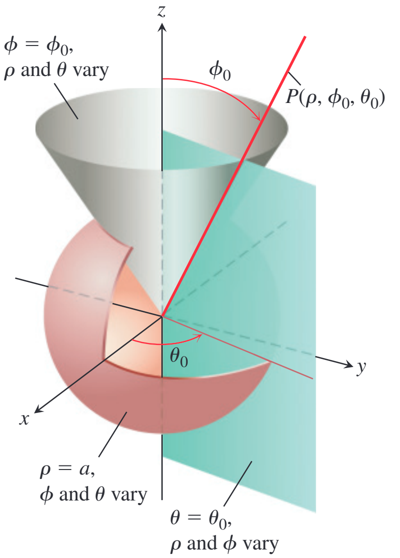


FIGURE 15.52 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$\begin{aligned} r &= \rho \sin \phi, & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\ z &= \rho \cos \phi, & y &= r \sin \theta = \rho \sin \phi \sin \theta, \end{aligned} \tag{1}$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

EXAMPLE 3 Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.

Solution We use Equations (1) to substitute for x, y, and z:

$$\begin{aligned} x^2 + y^2 + (z - 1)^2 &= 1 \\ \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 &= 1 \qquad \text{Eqs. (1)} \\ \rho^2 \sin^2 \phi \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{1} + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1 &= 1 \\ \rho^2 \underbrace{(\sin^2 \phi + \cos^2 \phi)}_{1} &= 2\rho \cos \phi \\ \rho^2 &= 2\rho \cos \phi \\ \rho &= 2 \cos \phi. \qquad \text{Includes } \rho = 0 \end{aligned}$$

The angle ϕ varies from 0 at the north pole of the sphere to $\pi/2$ at the south pole; the angle θ does not appear in the expression for ρ, reflecting the symmetry about the z-axis (see Figure 15.53). ■

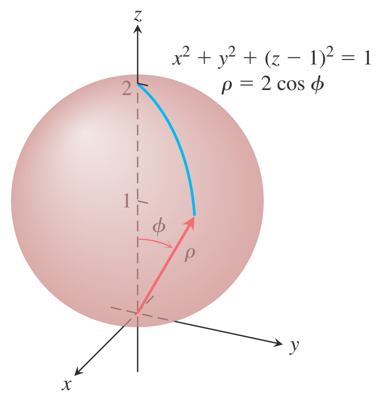


FIGURE 15.53 The sphere in Example 3.

EXAMPLE 4 Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.

Solution 1 *Use geometry.* The cone is symmetric with respect to the z-axis and cuts the first quadrant of the yz-plane along the line $z = y$. The angle between the cone and the positive z-axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$. (See Figure 15.54.)

Solution 2 *Use algebra.* If we use Equations (1) to substitute for x, y, and z, we obtain the same result:

$$\begin{aligned}
z &= \sqrt{x^2 + y^2} \\
\rho\cos\phi &= \sqrt{\rho^2\sin^2\phi} && \text{Example 3} \\
\rho\cos\phi &= \rho\sin\phi && \rho \ge 0,\ \sin\phi \ge 0 \\
\cos\phi &= \sin\phi && \text{Includes } \rho = 0 \text{ (the origin)} \\
\phi &= \frac{\pi}{4}. && 0 \le \phi \le \pi
\end{aligned}$$
■

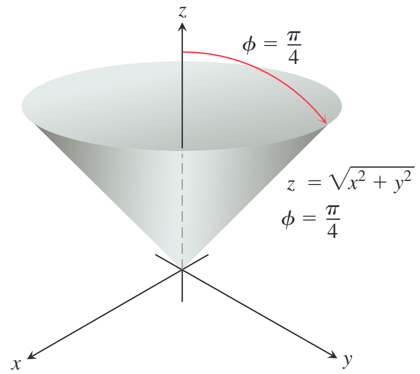


FIGURE 15.54 The cone in Example 4.

Spherical coordinates are useful for describing spheres centered at the origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$$\begin{aligned}
\rho &= 4 && \text{Sphere, radius 4, center at origin} \\
\phi &= \frac{2\pi}{3} && \text{Cone opening down from the origin, making an angle of } 2\pi/3 \text{ radians with the positive } z\text{-axis} \\
\theta &= \frac{\pi}{3}. && \text{Half-plane, hinged along the } z\text{-axis, making an angle of } \pi/3 \text{ radians with the positive } x\text{-axis}
\end{aligned}$$

When computing triple integrals over a solid region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by the changes $\Delta\rho_k$, $\Delta\phi_k$, and $\Delta\theta_k$ in ρ, ϕ, and θ. Such a spherical wedge has one edge a circular arc of length $\rho_k\,\Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k\,\Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a rectangular box of these dimensions when $\Delta\rho_k$, $\Delta\phi_k$, and $\Delta\theta_k$ are all small (Figure 15.55). It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$, a point chosen inside the wedge.

The corresponding Riemann sum for a function $f(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\,\rho_k^2 \sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k.$$

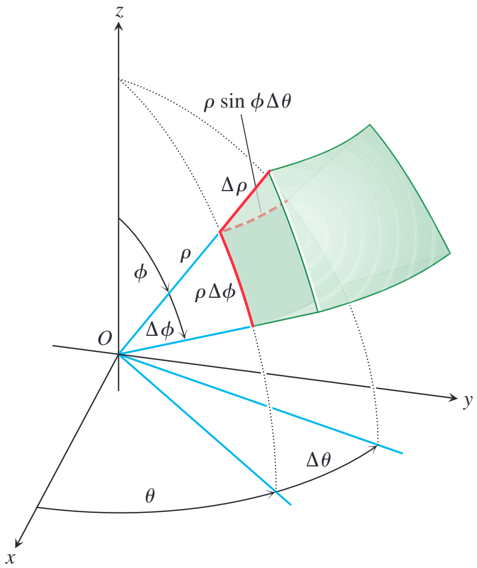


FIGURE 15.55 In spherical coordinates we use the volume of a spherical wedge, which closely approximates that of a rectangular box.

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the limit of the Riemann sums is the triple integral:

$$\boxed{\lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\,dV = \iiint_D f(\rho, \phi, \theta)\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.}$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is as follows. As with cylindrical coordinates, we restrict θ in the form $\alpha \leq \theta \leq \beta$ and $0 \leq \beta - \alpha \leq 2\pi$.

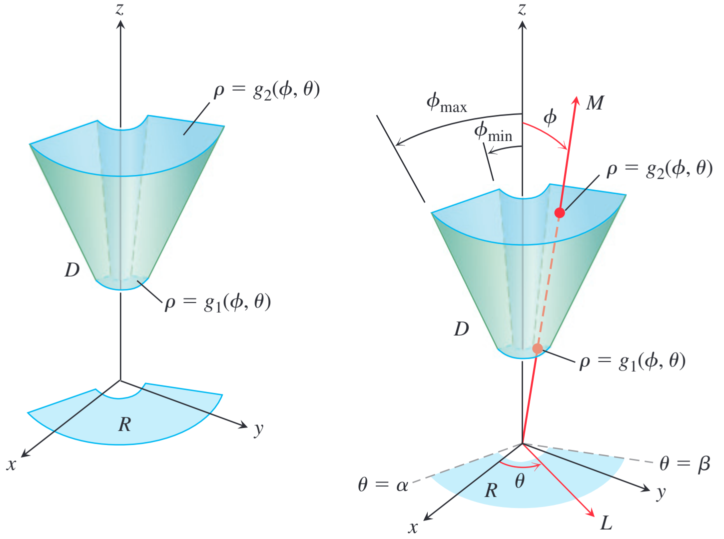
How to Integrate in Spherical Coordinates

To evaluate

$$\iiint_D f(\rho, \phi, \theta)\, dV$$

over a solid region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch*. Sketch the solid region D along with its projection R on the xy-plane. Label the surfaces that bound D.



2. *Find the ρ-limits of integration*. Draw a ray M from the origin through D, making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration shown in the above figure.
3. *Find the ϕ-limits of integration*. For any given θ, the angle ϕ that M makes with the positive z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. The ϕ-limits of integration may depend on θ, but they are often constant.
4. *Find the θ-limits of integration*. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

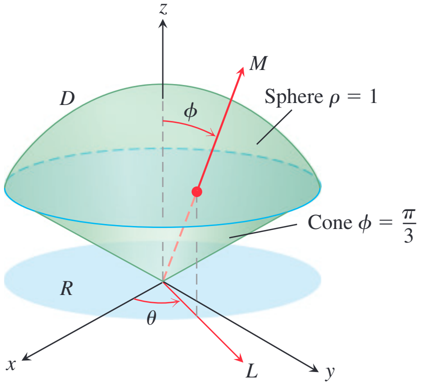


FIGURE 15.56 The ice cream cone in Example 5.

EXAMPLE 5 Find the volume of the "ice cream cone" D bounded above by the sphere $\rho = 1$ and bounded below by the cone $\phi = \pi/3$.

Solution The volume is $V = \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane (Figure 15.56).

The ρ-limits of integration. We draw a ray M from the origin through D, making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The ϕ-limits of integration. The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

The θ-limits of integration. The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=1} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$
$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

EXAMPLE 6 A solid of constant density $\delta = 1$ occupies the solid region D in Example 5. Find the solid's moment of inertia about the z-axis.

Solution In rectangular coordinates, the moment is

$$I_z = \iiint_D (x^2 + y^2) \, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint_D (\rho^2 \sin^2\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \iiint_D \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta.$$

For the region D in Example 5, this becomes

$$I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^5}{5}\right]_{\rho=0}^{\rho=1} \sin^3\phi \, d\phi \, d\theta$$
$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi/3} (1 - \cos^2\phi) \sin\phi \, d\phi \, d\theta = \frac{1}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_{\phi=0}^{\phi=\pi/3} d\theta$$
$$= \frac{1}{5} \int_0^{2\pi} \left(-\frac{1}{2} + \frac{1}{24} + 1 - \frac{1}{3}\right) d\theta = \frac{1}{5} \int_0^{2\pi} \frac{5}{24} \, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho \sin\phi \cos\theta$	$r = \rho \sin\phi$
$y = r\sin\theta$	$y = \rho \sin\phi \sin\theta$	$z = \rho \cos\phi$
$z = z$	$z = \rho \cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\, dy\, dz \\ &= r\, dz\, dr\, d\theta \\ &= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

In the next section we offer a more general procedure for determining dV in cylindrical and spherical coordinates. The results, of course, will be the same.

EXERCISES 15.7

In Exercises 1–12, sketch the region described by the following cylindrical coordinates in three-dimensional space.

1. $r = 2$
2. $\theta = \frac{\pi}{4}$
3. $z = -1$
4. $z = r$
5. $r = \theta$
6. $z = r\sin\theta$
7. $r^2 + z^2 = 4$
8. $1 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{3}$
9. $r \le z \le \sqrt{9 - r^2}$
10. $0 \le r \le 2\sin\theta, \quad 1 \le z \le 3$
11. $0 \le r \le 4\cos\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le z \le 5$
12. $0 \le r \le 3, \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le z \le r\cos\theta$

In Exercises 13–22, sketch the region described by the following spherical coordinates in three-dimensional space.

13. $\rho = 3$
14. $\phi = \frac{\pi}{6}$
15. $\theta = \frac{2}{3}\pi$
16. $\rho = \csc\phi$
17. $\rho\cos\phi = 4$
18. $1 \le \rho \le 2\sec\phi, \quad 0 \le \phi \le \frac{\pi}{4}$
19. $0 \le \rho \le 3\csc\phi$
20. $0 \le \rho \le 1, \quad \frac{\pi}{2} \le \phi \le \pi, \quad 0 \le \theta \le \pi$
21. $0 \le \rho\cos\theta\sin\phi \le 2, \quad 0 \le \rho\sin\theta\sin\phi \le 3,$ $0 \le \rho\cos\phi \le 4$
22. $4\sec\phi \le \rho \le 5, 0 \le \phi \le \frac{\pi}{2}$

Evaluating Integrals in Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 23–28.

23. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$
24. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} r\,dz\,dr\,d\theta$
25. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} r\,dz\,dr\,d\theta$
26. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta$
27. $\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,r\,dz\,dr\,d\theta$
28. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2} (r^2\sin^2\theta + z^2)\,r\,dz\,dr\,d\theta$

Changing the Order of Integration in Cylindrical Coordinates

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 29–32.

29. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta$
30. $\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dr\,d\theta\,dz$
31. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi} (r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz$
32. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi} (r\sin\theta + 1)\,r\,d\theta\,dz\,dr$
33. Let D be the solid region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
    a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$
34. Let D be the solid region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
    a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

Finding Iterated Integrals in Cylindrical Coordinates

35. Give the limits of integration for evaluating the integral

$$\iiint_D f(r, \theta, z)\,r\,dz\,dr\,d\theta$$

as an iterated integral over the solid region D that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

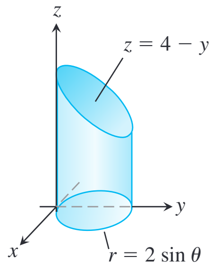
36. Convert the integral

$$\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_0^x (x^2 + y^2)\,dz\,dx\,dy$$

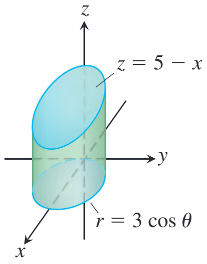
to an equivalent integral in cylindrical coordinates and evaluate the result.

In Exercises 37–42, set up the iterated integral for evaluating $\iiint_D f(r, \theta, z)\,r\,dz\,dr\,d\theta$ over the given solid region D.

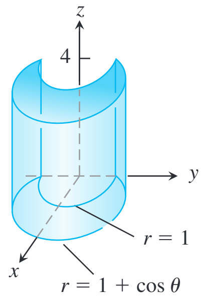
37. D is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



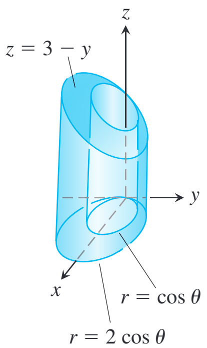
38. D is the right circular cylinder whose base is the circle $r = 3\cos\theta$ and whose top lies in the plane $z = 5 - x$.



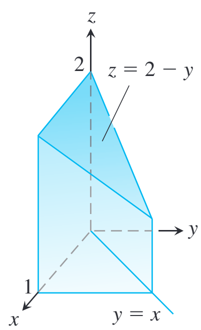
39. D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



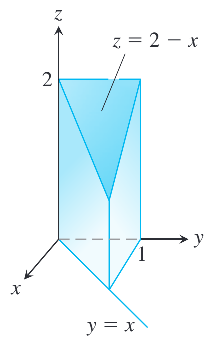
40. D is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.



41. D is the right prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 2 - y$.



42. D is the right prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.



Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 43–48.

43. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

44. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

45. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

46. $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3\sin^3\phi\, d\rho\, d\phi\, d\theta$

47. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

48. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

Changing the Order of Integration in Spherical Coordinates

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders give the same value and are occasionally easier to evaluate. Evaluate the integrals in Exercises 49–52.

49. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3\sin 2\phi\, d\phi\, d\theta\, d\rho$

50. $\displaystyle\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2\sin\phi\, d\theta\, d\rho\, d\phi$

51. $\displaystyle\int_0^{1}\int_0^{\pi}\int_0^{\pi/4} 12\rho\sin^3\phi\, d\phi\, d\theta\, d\rho$

52. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi\, d\rho\, d\theta\, d\phi$

53. Let D be the region in Exercise 33. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho\, d\phi\, d\theta$

b. $d\phi\, d\rho\, d\theta$

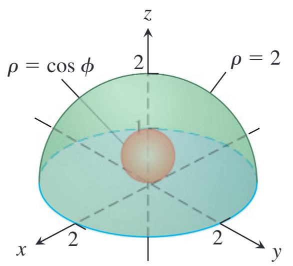
54. Let D be the solid region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho\, d\phi\, d\theta$ **b.** $d\phi\, d\rho\, d\theta$

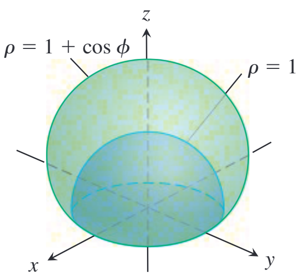
Finding Iterated Integrals in Spherical Coordinates

In Exercises 55–60, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

55. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geq 0$



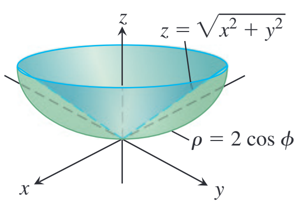
56. The solid bounded below by the hemisphere $\rho = 1$, $z \geq 0$, and above by the surface $\rho = 1 + \cos\phi$



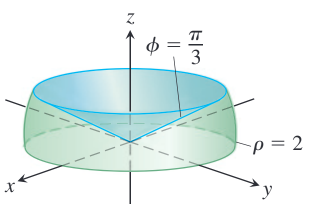
57. The solid enclosed by the surface $\rho = 1 - \cos\phi$

58. The upper portion cut from the solid in Exercise 57 by the xy-plane

59. The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$



60. The solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$



Finding Triple Integrals

61. Set up triple integrals for the volume of the sphere $\rho = 2$ in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates.

62. Let D be the solid region in the first octant that is bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$. Express the volume of D as an iterated triple integral in **(a)** cylindrical and **(b)** spherical coordinates. Then **(c)** find the volume.

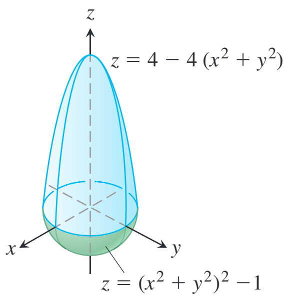
63. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates. Then **(d)** find the volume by evaluating one of the three triple integrals.

64. Let D be the solid hemisphere $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$. If the density is $\delta(x, y, z) = 1$, express the moment of intertia I_z as an iterated integral in **(a)** cylindrical and **(b)** spherical coordinates. Then **(c)** find I_z.

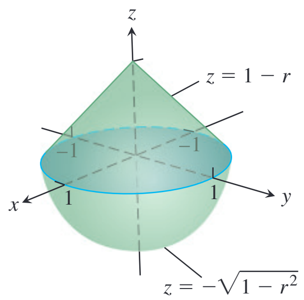
Volumes

Find the volumes of the solids in Exercises 65–70.

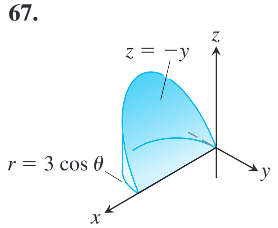
65.



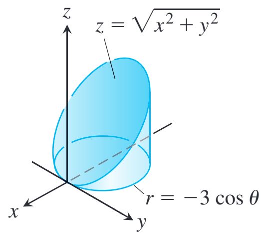
66.



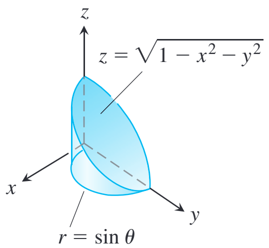
67.



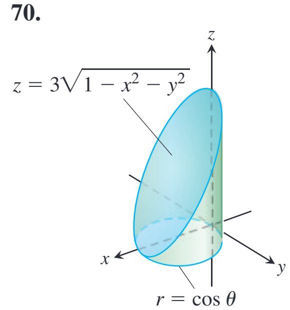
68.



69.



70.



71. Ball and cones Find the volume of the portion of the ball $\rho \leq a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.

72. Ball and half-planes Find the volume of the region cut from the ball $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

73. **Ball and plane** Find the volume of the smaller region cut from the ball $\rho \le \sqrt{2}$ by the plane $z = 1$.

74. **Cone and planes** Find the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

75. **Cylinder and paraboloid** Find the volume of the solid region bounded below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

76. **Cylinder and paraboloids** Find the volume of the solid region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

77. **Cylinder and cones** Find the volume of the solid cut from the thick-walled cylinder $1 \le x^2 + y^2 \le 2$ by the cones $z = \pm\sqrt{x^2 + y^2}$.

78. **Sphere and cylinder** Find the volume of the solid region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

79. **Cylinder and planes** Find the volume of the solid region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

80. **Cylinder and planes** Find the volume of the solid region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.

81. **Region trapped by paraboloids** Find the volume of the solid region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

82. **Paraboloid and cylinder** Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$, bounded below by the xy-plane, and lying *outside* the cylinder $x^2 + y^2 = 1$.

83. **Cylinder and sphere** Find the volume of the region cut from the solid cylinder $x^2 + y^2 \le 1$ by the sphere $x^2 + y^2 + z^2 = 4$.

84. **Sphere and paraboloid** Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Average Values

85. Find the average value of the function $f(r, \theta, z) = r$ over the solid region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.

86. Find the average value of the function $f(r, \theta, z) = r$ over the solid ball bounded by the sphere $r^2 + z^2 = 1$. (This is the sphere $x^2 + y^2 + z^2 = 1$.)

87. Find the average value of the function $f(\rho, \phi, \theta) = \rho$ over the solid ball $\rho \le 1$.

88. Find the average value of the function $f(\rho, \phi, \theta) = \rho\cos\phi$ over the upper half of the solid ball $\rho \le 1$, $0 \le \phi \le \pi/2$.

Masses, Moments, and Centroids

89. **Center of mass** A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r$, $r \ge 0$, and on the sides by the cylinder $r = 1$. Find the center of mass.

90. **Centroid** Find the centroid of the solid region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.

91. **Centroid** Find the centroid of the solid in Exercise 60.

92. **Centroid** Find the centroid of the solid bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/4$.

93. **Centroid** Find the centroid of the solid region that is bounded above by the surface $z = \sqrt{r}$, on the sides by the cylinder $r = 4$, and below by the xy-plane.

94. **Centroid** Find the centroid of the region cut from the solid ball $r^2 + z^2 \le 1$ by the half-planes $\theta = -\pi/3$, $r \ge 0$, and $\theta = \pi/3$, $r \ge 0$.

95. **Moment of inertia of solid cone** Find the moment of inertia of a solid right circular cone of base radius 1 and height 1 about an axis through the vertex parallel to the base if the density is $\delta = 1$.

96. **Moment of inertia of ball** Find the moment of inertia of a ball of radius a about a diameter if the density is $\delta = 1$.

97. **Moment of inertia of solid cone** Find the moment of inertia of a solid right circular cone of base radius a and height h about its axis if the density is $\delta = 1$. (*Hint:* Place the cone with its vertex at the origin and its axis along the z-axis.)

98. **Variable density** A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $r = 1$. Find the center of mass and the moment of inertia about the z-axis if the density is

   a. $\delta(r, \theta, z) = z$

   b. $\delta(r, \theta, z) = r$.

99. **Variable density** A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia about the z-axis if the density is

   a. $\delta(r, \theta, z) = z$

   b. $\delta(r, \theta, z) = z^2$.

100. **Variable density** A solid ball is bounded by the sphere $\rho = a$. Find the moment of inertia about the z-axis if the density is

   a. $\delta(\rho, \phi, \theta) = \rho^2$

   b. $\delta(\rho, \phi, \theta) = r = \rho\sin\phi$.

101. **Centroid of solid semi-ellipsoid** Show that the centroid of the solid semi-ellipsoid of revolution $(r^2/a^2) + (z^2/h^2) \le 1$, $z \ge 0$, lies on the z-axis three-eighths of the way from the base to the top. The special case $h = a$ gives a solid hemisphere. Thus, the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base to the top.

102. **Centroid of solid cone** Show that the centroid of a solid right circular cone is one-fourth of the way from the base to the vertex. (In general, the centroid of a solid cone or pyramid is one-fourth of the way from the centroid of the base to the vertex.)

103. **Density of center of a planet** A planet is in the shape of a sphere of radius R and total mass M with spherically symmetric density distribution that increases linearly as one approaches its center. What is the density at the center of this planet if the density at its edge (surface) is taken to be zero?

104. **Mass of planet's atmosphere** A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

Theory and Examples

105. Vertical planes in cylindrical coordinates

a. Show that planes perpendicular to the x-axis have equations of the form $r = a \sec \theta$ in cylindrical coordinates.

b. Show that planes perpendicular to the y-axis have equations of the form $r = b \csc \theta$.

106. (*Continuation of Exercise 105.*) Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c$, $c \neq 0$.

107. Symmetry What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

108. Symmetry What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

15.8 Substitutions in Multiple Integrals

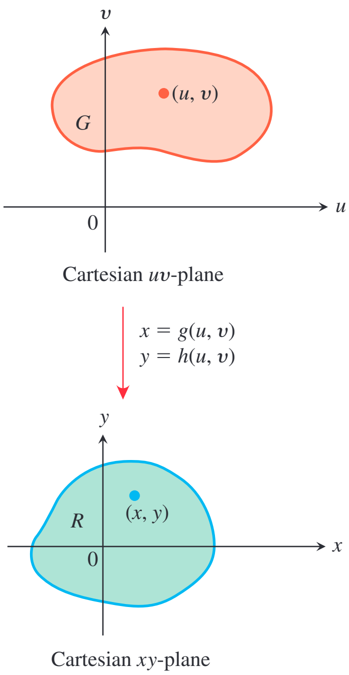


FIGURE 15.57 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy-plane into an integral over a region G in the uv-plane.

This section introduces the ideas involved in coordinate transformations to evaluate multiple integrals by substitution. The method replaces complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both. A thorough discussion of multivariable transformations and substitutions is best left to a more advanced course, but our introduction here shows how the substitutions just studied reflect the general idea derived for single integral calculus.

Substitutions in Double Integrals

The polar coordinate substitution of Section 15.4 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv-plane is transformed into the region R in the xy-plane by equations of the form

$$x = g(u, v), \qquad y = h(u, v),$$

as suggested in Figure 15.57. We assume the transformation is one-to-one on the interior of G. We call R the **image** of G under the transformation, and G the **preimage** of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G?

To gain some insight into the question, we look again at the single variable case. To be consistent with how we are using them now, we interchange the variables x and u used in the substitution method for single integrals in Chapter 5, so the equation is

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du. \qquad x = g(u),\ dx = g'(u)du$$

To propose an analogue for substitution in a double integral $\iint_R f(x, y)\,dx\,dy$, we need a derivative factor like $g'(u)$ as a multiplier that transforms the area element $du\,dv$ in the region G to its corresponding area element $dx\,dy$ in the region R. We denote this factor by J. In continuing with our analogy, it is reasonable to assume that J is a function of both variables u and v, just as g' is a function of the single variable u. Moreover, J should register instantaneous change, so partial derivatives are going to be involved in its expression. Since four partial derivatives are associated with the transforming equations $x = g(u, v)$ and $y = h(u, v)$, it is also reasonable to assume that the factor $J(u, v)$ we seek includes them all. These features are captured in the following definition, which is constructed from the partial derivatives and is named after the German mathematician Carl Jacobi.

HISTORICAL BIOGRAPHY
Carl Gustav Jacob Jacobi (1804–1851)
www.bit.ly/2xZS8Wi