COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

a. Plot the Cartesian region of integration in the xy-plane.

b. Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ.

c. Using the results in part (b), plot the polar region of integration in the $r\theta$-plane.

d. Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using the CAS integration utility.

49. $\int_0^1 \int_x^1 \frac{y}{x^2 + y^2}\, dy\, dx$

50. $\int_0^1 \int_0^{x/2} \frac{x}{x^2 + y^2}\, dy\, dx$

51. $\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2 + y^2}}\, dx\, dy$

52. $\int_0^1 \int_y^{2-y} \sqrt{x + y}\, dx\, dy$

15.5 Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

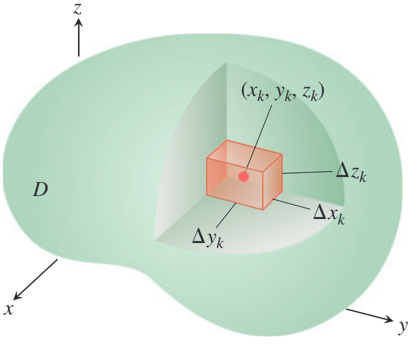


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded solid region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes (Figure 15.30). We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k. \tag{1}$$

We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k, Δy_k, Δz_k, and the norm of the partition $\|P\|$, the largest value among Δx_k, Δy_k, Δz_k, all approach zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is **integrable** over D. As before, it can be shown that when F is continuous and the bounding surface of D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable. In this case, as $\|P\| \to 0$ and the number of cells n goes to ∞, the sums S_n approach a limit. We call this limit the **triple integral of F over D** and write

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\, dV \quad \text{or} \quad \lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\, dx\, dy\, dz.$$

The regions D over which continuous functions are integrable are those having "reasonably smooth" boundaries.

Volume of a Solid Region in Space

If F is the constant function whose value is 1, then the sums in Equation (1) reduce to

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k = \sum_{k=1}^{n} 1 \cdot \Delta V_k = \sum_{k=1}^{n} \Delta V_k.$$

As Δx_k, Δy_k, and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV.$$

DEFINITION The **volume** of a closed and bounded solid region D in space is

$$V = \iiint_D dV.$$

This definition is in agreement with our previous definitions of volume, although we omit the verification of this fact. As we will see in a moment, this integral enables us to calculate the volumes of solids enclosed by curved surfaces. These are more general solids than the ones encountered before (Chapter 6 and Section 15.2).

Iterated Integrals

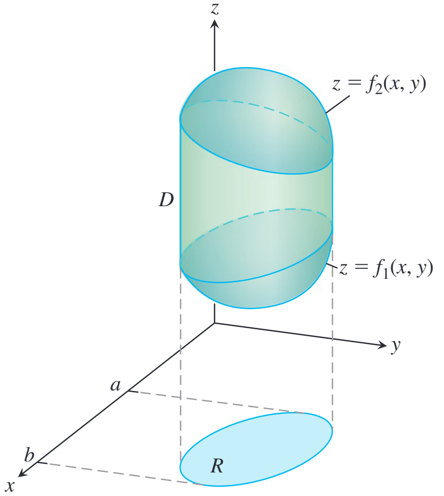
We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.2) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these iterated integrals.

To evaluate

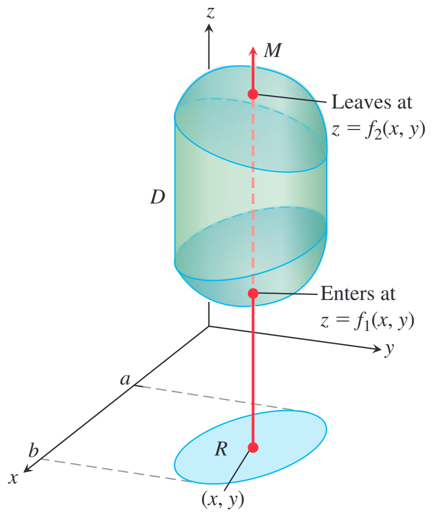
$$\iiint_D F(x, y, z)\, dV$$

over a solid region D, integrate first with respect to z, then with respect to y, and finally with respect to x. (You might choose a different order of integration, but the procedure is similar, as we illustrate in Example 2.)

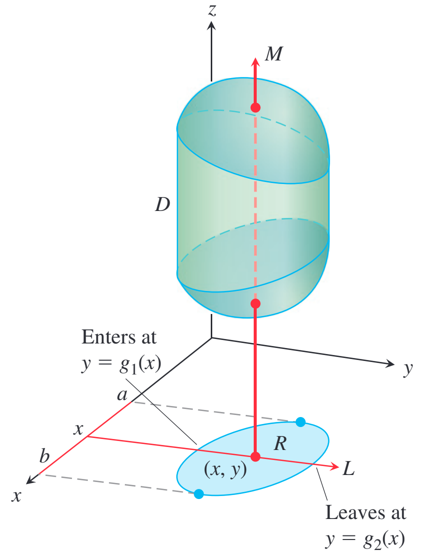
1. *Sketch.* Sketch the solid region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of the solid region D lies in the plane of the last two variables with respect to which the iterated integration takes place. The limits of an iterated triple integral satisfy these properties:

- The limits of the *outside* integral are *constants* (they do not depend on any of the three variables of integration),
- the limits of the *middle* integral are *functions* that may depend on the variable of the *outside* integral, and
- the limits of the *inside* integral are *functions* that may depend on two variables: the *middle* integration variable and the *outside* integration variable.

The preceding procedure applies whenever a solid region D is bounded above and below by a surface, and when the "shadow" region R is bounded by a lower and upper curve. It does not apply to regions with more complicated shapes (such as regions containing holes); although, sometimes such regions can be subdivided into simpler regions for which the procedure does apply.

We illustrate this method of finding the limits of integration in our first example.

EXAMPLE 1 Let S be the sphere of radius 5 centered at the origin, and let D be the solid region under the sphere that lies above the plane $z = 3$. Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the region D.

Solution The solid region under the sphere that lies above the plane $z = 3$ is enclosed by the surfaces $x^2 + y^2 + z^2 = 25$ and $z = 3$.

To find the limits of integration, we first sketch the solid region, as shown in Figure 15.31. The "shadow region" R in the xy-plane is a circle of some radius centered at the origin. By considering a side view of the region D, we can determine that the radius of this circle is 4; see Figure 15.32a.

If we fix a point (x, y) in R and draw a vertical line M above (x, y), then we see that this line enters the region D at the height $z = 3$ and leaves the region at the height $z = \sqrt{25 - x^2 - y^2}$; see Figure 15.31. This gives us the z-limits of integration.

To find the y-limits of integration, we consider a line L that lies in the region R, passes through the point (x, y), and is parallel to the y-axis. For clarity we have separately pictured the region R and the line L in Figure 15.32b. The line L enters R when $y = -\sqrt{16 - x^2}$ and exits when $y = \sqrt{16 - x^2}$. This gives us the y-limits of integration.

Finally, as L sweeps across R from left to right, the value of x varies from $x = -4$ to $x = 4$. This gives us the x-limits of integration. Therefore, the triple integral of F over the region D is given by

$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} F(x, y, z)\, dz\, dy\, dx. \quad \blacksquare$$

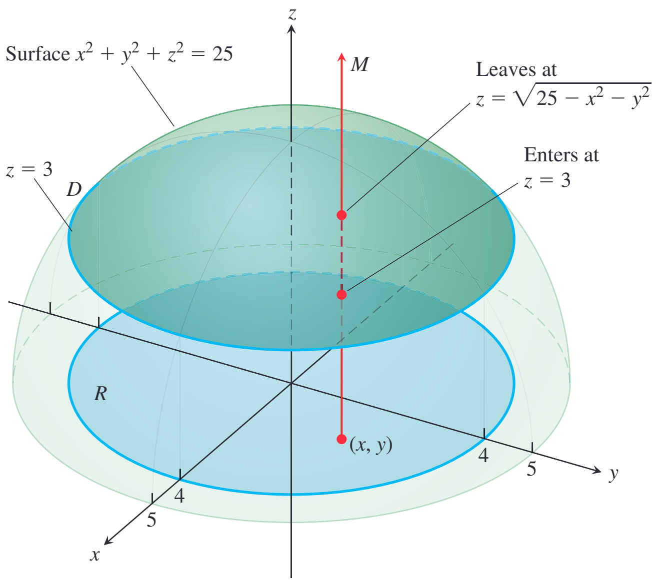


FIGURE 15.31 Finding the limits of integration for evaluating the triple integral of a function defined over the portion of the sphere of radius 5 that lies above the plane $z = 3$ (Example 1).

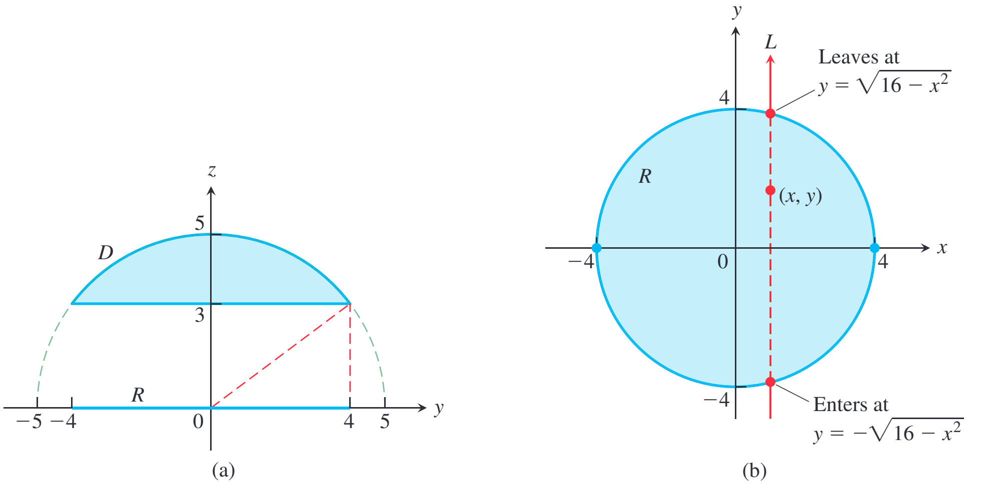


FIGURE 15.32 (a) Side view of the solid region from Example 1, looking down the x-axis. The dashed right triangle has a hypotenuse of length 5 and sides of lengths 3 and 4. In this side view, the shadow region R lies between -4 and 4 on the y-axis. (b) The "shadow region" R shown face-on in the xy-plane.

The region D in Example 1 has a great deal of symmetry, which makes visualization easier. Even without symmetry, the steps in finding the limits of integration are the same, as shown in the next example.

EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D whose vertices are $O(0,0,0)$, $A(1,1,0)$, $B(0,1,0)$, and $C(0,1,1)$. Use the order of integration $dz\,dy\,dx$.

Solution The solid region D and its "shadow" R in the xy-plane are shown in Figure 15.33a. The "top" face is contained in the plane through the points O, A, and C. Following the procedure introduced in Example 7 of Section 12.5, we first form a normal vector to that plane:

$$\mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k},$$

and then use this vector and the coordinates of O to set up an equation for the plane:

$$1(x - 0) - 1(y - 0) + 1(z - 0) = 0$$
$$x - y + z = 0.$$

The "side" face of D is parallel to the xz-plane, the "back" face lies in the yz-plane, and the "bottom" face is contained in the xy-plane.

To find the z-limits of integration, fix a point (x, y) in the shadow region R, and consider the vertical line M that passes through (x, y) and is parallel to the z-axis. This line enters D at the height $z = 0$, and it exits at height $z = y - x$.

To find the y-limits of integration we again fix a point (x, y) in R, but now we consider a line L that lies in R, passes through (x, y), and is parallel to the y-axis. This line is shown in Figure 15.33a and also in the face-on view of R that is pictured in Figure 15.33b. The line L enters R when $y = x$ and exits when $y = 1$.

Finally, as L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. Therefore, the triple integral of F over the region D is given by

$$\iiint\limits_D F(x, y, z)\,dz\,dy\,dx = \int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\,dz\,dy\,dx. \quad \blacksquare$$

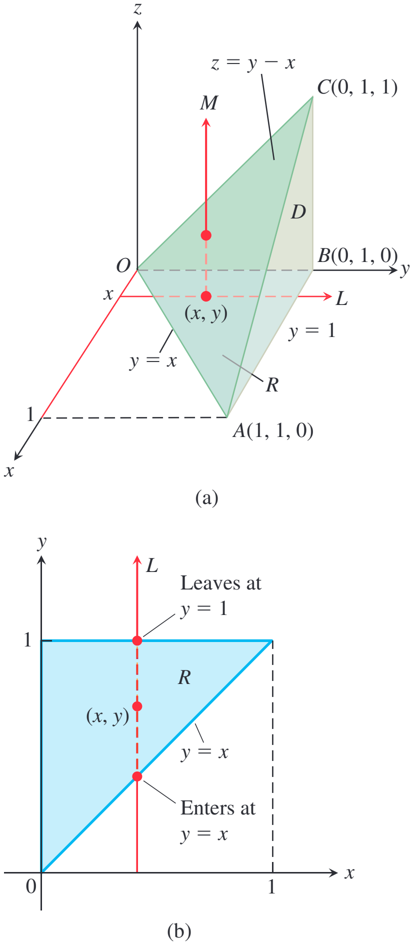


FIGURE 15.33 (a) The tetrahedron in Example 2, showing how the limits of integration are found for the order $dz\,dy\,dx$. (b) The "shadow region" R shown face-on in the xy-plane.

In the next example we project the region D onto the xz-plane instead of the xy-plane, to show how to use a different order of integration.

EXAMPLE 3 Find the volume of the tetrahedron D from Example 2 by integrating $F(x, y, z) = 1$ over the region using the order $dz\, dy\, dx$. Then do the same calculation using the order $dy\, dz\, dx$.

Solution Using the limits of integration that we found in Example 2, we calculate the volume of the tetrahedron as follows:

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx \qquad \text{Integrand is 1 when computing volume.}$$

$$= \int_0^1 \int_x^1 (y - x)\, dy\, dx \qquad \text{Integrate over } z \text{ and evaluate.}$$

$$= \int_0^1 \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=1} dx \qquad \text{Integrate over } y.$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx \qquad \text{Evaluate.}$$

$$= \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 \qquad \text{Integrate over } x.$$

$$= \frac{1}{6}. \qquad \text{Evaluate.}$$

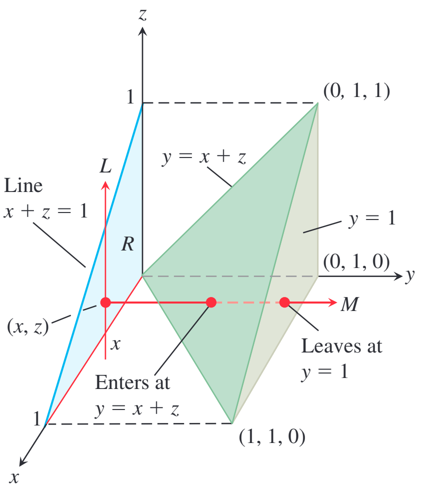


FIGURE 15.34 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Example 3).

Now we will compute the volume using the order of integration $dy\, dz\, dx$. The procedure for finding the limits of integration is similar, except that we find the limits for y first, then for z, and then for x. The region D is the same tetrahedron as before, but now the "shadow region" R lies in the xz-plane, as shown in Figure 15.34.

To find the y-limits of integration, we fix a point (x, z) in the shadow R and consider the line M that passes through (x, z) and is parallel to the y-axis. As shown in Figure 15.34, this line enters D when $y = x + z$, and it leaves when $y = 1$.

Next we find the z-limits of integration. The line L that passes through a point (x, z) in R and is parallel to the z-axis enters R when $z = 0$ and exits when $z = 1 - x$ (see Figure 15.34).

Finally, as L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. Therefore, the volume of the tetrahedron is

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx$$

$$= \int_0^1 \left[(1 - x)z - \frac{1}{2}z^2\right]_{z=0}^{z=1-x} dx$$

$$= \int_0^1 \left[(1 - x)^2 - \frac{1}{2}(1 - x)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 (1 - x)^2\, dx$$

$$= -\frac{1}{6}(1 - x)^3\Big]_0^1 = \frac{1}{6}.$$ ■

Next we set up and evaluate a triple integral over a more complicated region.

EXAMPLE 4 Find the volume of the solid region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.35) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

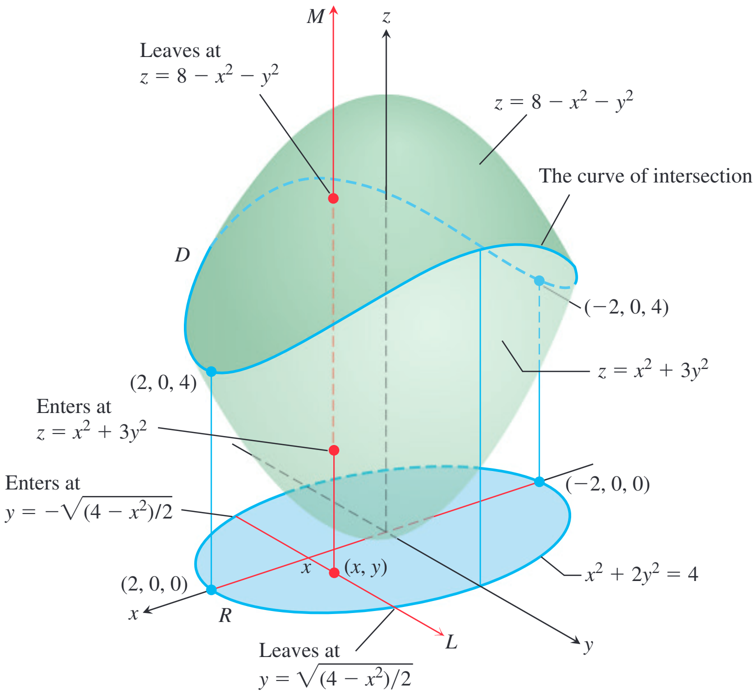


FIGURE 15.35 The volume of the region enclosed by two paraboloids, calculated in Example 4.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Next we find the y-limits of integration. The line L through (x, y) that lies parallel to the y-axis enters the region R when $y = -\sqrt{(4 - x^2)/2}$ and leaves when $y = \sqrt{(4 - x^2)/2}$.

Finally, we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$V = \iiint_D dz\, dy\, dx \qquad \text{Integrand is 1 when computing volume.}$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \qquad \text{Form an iterated integral.}$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \qquad \text{Integrate over } z \text{ and evaluate.}$$

$$= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \qquad \text{Integrate over } y.$$

$$= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right) dx \qquad \text{Evaluate.}$$

$$= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx$$

$$= \frac{4\sqrt{2}}{3}\int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin\theta$$

Average Value of a Function in Space

The average value of a function F over a solid region D in space is defined by the formula

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D}\iiint_D F\, dV. \qquad (2)$$

For example, if $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then the average value of F over D is the average distance of points in D from the origin. If $F(x, y, z)$ is the temperature at (x, y, z) on a solid that occupies a region D in space, then the average value of F over D is the average temperature of the solid.

EXAMPLE 5 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution We sketch the cube with enough detail to show the limits of integration (Figure 15.36). We then use Equation (2) to calculate the average value of F over the cube.

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2\int_0^2\int_0^2 xyz\, dx\, dy\, dz = \int_0^2\int_0^2\left[\frac{x^2}{2}yz\right]_{x=0}^{x=2} dy\, dz = \int_0^2\int_0^2 2yz\, dy\, dz$$

$$= \int_0^2\left[y^2z\right]_{y=0}^{y=2} dz = \int_0^2 4z\, dz = \left[2z^2\right]_0^2 = 8.$$

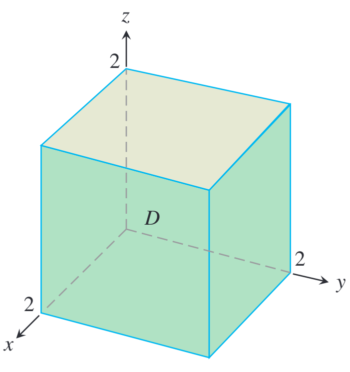


FIGURE 15.36 The region of integration in Example 5.

With these values, Equation (2) gives

$$\begin{array}{c}\text{Average value of}\\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}}\iiint_{\text{cube}} xyz\, dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx\, dy\, dz$, but any of the other five possible orders would have done as well.

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals. Simply replace the double integrals in the four properties given in Section 15.2, page 904, with triple integrals.

EXERCISES 15.5

Triple Integrals in Different Iteration Orders

1. Evaluate the integral in Example 3, taking $F(x, y, z) = 1$ to find the volume of the tetrahedron in the order $dz\,dx\,dy$.

2. Volume of rectangular solid Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.

3. Volume of tetrahedron Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.

4. Volume of solid Write six different iterated triple integrals for the volume of the solid region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.

5. Volume enclosed by paraboloids Let D be the solid region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

6. Volume inside paraboloid beneath a plane Let D be the solid region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\,dx\,dy$ and $dz\,dy\,dx$ that give the volume of D. Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

8. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

9. $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz$

10. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\,dy\,dx$

11. $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z\,dx\,dy\,dz$

12. $\int_{-1}^1 \int_0^1 \int_0^2 (x + y + z)\,dy\,dx\,dz$

13. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

14. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\,dx\,dy$

15. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\,dy\,dx$

16. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\,dz\,dy\,dx$

17. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u + v + w)\,du\,dv\,dw$ (uvw-space)

18. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e se^s \ln r \frac{(\ln t)^2}{t}\,dt\,dr\,ds$ (rst-space)

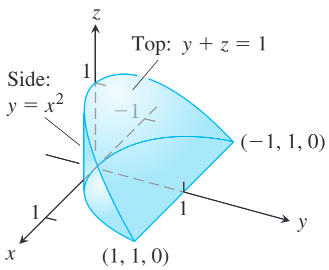
19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x\,dx\,dt\,dv$ (tvx-space)

20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr$ (pqr-space)

Finding Equivalent Iterated Integrals

21. Here is the region of integration of the integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\,dy\,dx.$$

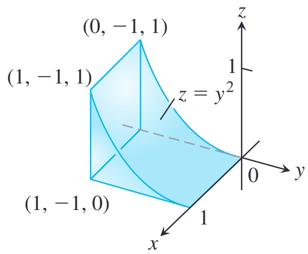


Rewrite the integral as an equivalent iterated integral in the order

a. $dy\,dz\,dx$ **b.** $dy\,dx\,dz$
c. $dx\,dy\,dz$ **d.** $dx\,dz\,dy$
e. $dz\,dx\,dy$.

22. Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\,dy\,dx$$



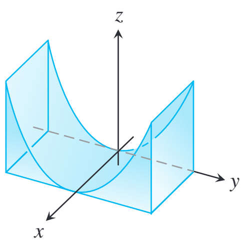
Rewrite the integral as an equivalent iterated integral in the order

a. $dy\,dz\,dx$ **b.** $dy\,dx\,dz$
c. $dx\,dy\,dz$ **d.** $dx\,dz\,dy$
e. $dz\,dx\,dy$.

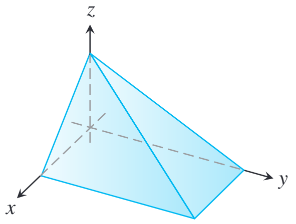
Finding Volumes Using Triple Integrals

Find the volumes of the solid regions in Exercises 23–36.

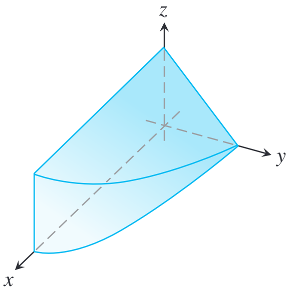
23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$



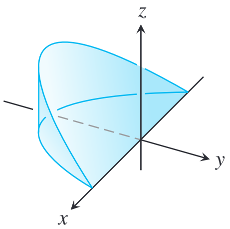
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



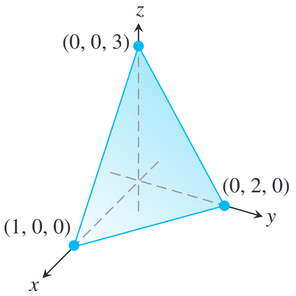
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



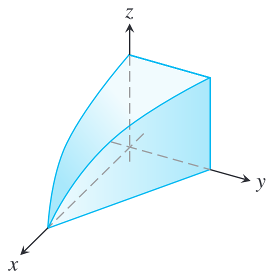
26. The wedge cut from the cylinder $x^2 + y^2 = 1$ with $z \geq 0$ by the planes $z = -y$ and $z = 0$



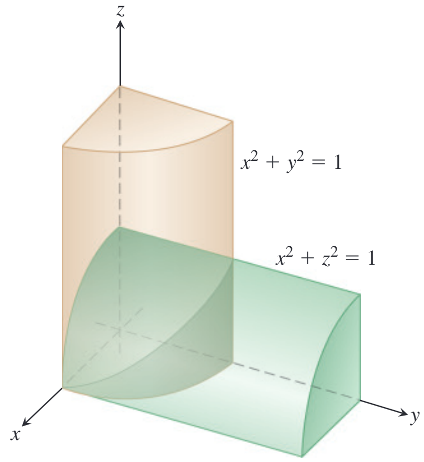
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$



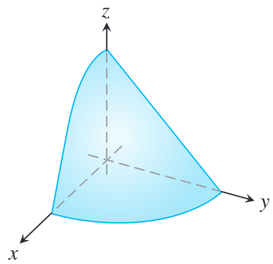
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \leq x \leq 1$



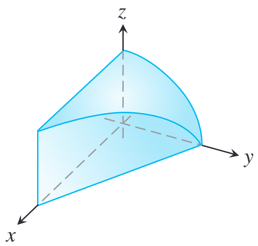
29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure



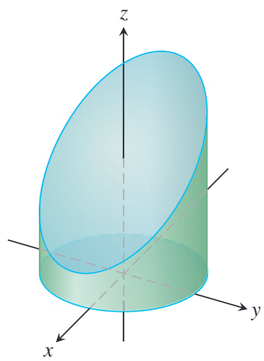
30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant

34. The finite region bounded by the planes $z = x$, $x + z = 8$, $z = y$, $y = 8$, and $z = 0$

35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \le 4$ by the xy-plane and the plane $z = x + 2$

36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy-plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular box in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\displaystyle\int_0^4\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$

42. $\displaystyle\int_0^1\int_0^1\int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz$

43. $\displaystyle\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin \pi y^2}{y^2}\,dx\,dy\,dz$

44. $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx$

Theory and Examples

45. Finding an upper limit of an iterated integral Solve for a:

$$\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15}.$$

46. Ellipsoid For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π?

47. Minimizing a triple integral What domain D in space minimizes the value of the integral

$$\iiint_D (4x^2 + 4y^2 + z^2 - 4)\,dV?$$

Give reasons for your answer.

48. Maximizing a triple integral What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2)\,dV?$$

Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS integration utility to evaluate the triple integral of the given function over the specified solid region.

49. $F(x, y, z) = x^2y^2z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$

50. $F(x, y, z) = |xyz|$ over the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$

51. $F(x, y, z) = \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$ over the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$

52. $F(x, y, z) = x^4 + y^2 + z^2$ over the solid sphere $x^2 + y^2 + z^2 \le 1$

15.6 Applications

This section shows how to calculate the masses and moments of two- and three-dimensional objects in Cartesian coordinates. The definitions and ideas are similar to the single-variable case we studied in Section 6.6, but now we can consider more general situations.

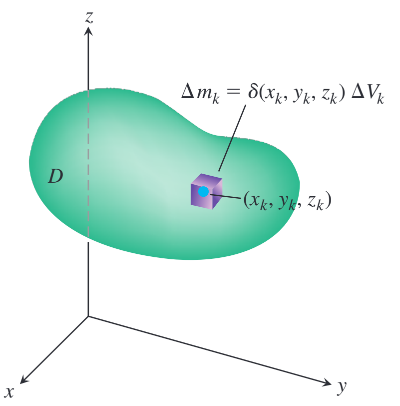


FIGURE 15.37 To define an object's mass, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

Masses and First Moments

If $\delta(x, y, z)$ is the density (mass per unit volume) of an object occupying a solid region D in space, the integral of δ over D gives the **mass** of the object. To see why, imagine partitioning the object into n mass elements like the one in Figure 15.37. The object's mass is the limit

$$M = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta m_k = \lim_{n\to\infty}\sum_{k=1}^{n}\delta(x_k, y_k, z_k)\,\Delta V_k = \iiint_D \delta(x, y, z)\,dV.$$

The *first moment* of a solid region D about a coordinate plane is defined as the triple integral over D of the (signed) distance from a point (x, y, z) in D to the plane multiplied