19. $\iint_R xy\cos y\, dA, \quad R: \; -1 \le x \le 1, \quad 0 \le y \le \pi$

20. $\iint_R y\sin(x+y)\, dA, \quad R: \; -\pi \le x \le 0, \quad 0 \le y \le \pi$

21. $\iint_R e^{x-y}\, dA, \quad R: \; 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

22. $\iint_R xye^{xy^2}\, dA, \quad R: \; 0 \le x \le 2, \quad 0 \le y \le 1$

23. $\iint_R \frac{xy^3}{x^2+1}\, dA, \quad R: \; 0 \le x \le 1, \quad 0 \le y \le 2$

24. $\iint_R \frac{y}{x^2y^2+1}\, dA, \quad R: \; 0 \le x \le 1, \quad 0 \le y \le 1$

In Exercises 25 and 26, integrate f over the given region.

25. Square $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

26. Rectangle $f(x, y) = y\cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

In Exercises 27 and 28, sketch the solid whose volume is given by the specified integral.

27. $\int_0^1\int_0^2 (9 - x^2 - y^2)\, dy\, dx$ **28.** $\int_0^3\int_1^4 (7 - x - y)\, dx\, dy$

29. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

30. Find the volume of the region bounded above by the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: \; 0 \le x \le 2$, $0 \le y \le 2$.

31. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1$, $0 \le y \le 1$.

32. Find the volume of the region bounded above by the plane $z = y/2$ and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$.

33. Find the volume of the region bounded above by the surface $z = 2\sin x\cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

34. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$, $0 \le y \le 2$.

35. Find a value of the constant k so that $\int_1^2\int_0^3 kx^2y\, dx\, dy = 1$.

36. Evaluate $\int_{-1}^1\int_0^{\pi/2} x\sin\sqrt{y}\, dy\, dx$.

37. Use Fubini's Theorem to evaluate

$$\int_0^2\int_0^1 \frac{x}{1+xy}\, dx\, dy.$$

38. Use Fubini's Theorem to evaluate

$$\int_0^1\int_0^3 xe^{xy}\, dx\, dy.$$

T **39.** Use a software application to compute the integrals

a. $\int_0^1\int_0^2 \frac{y-x}{(x+y)^3}\, dx\, dy$ **b.** $\int_0^2\int_0^1 \frac{y-x}{(x+y)^3}\, dy\, dx$

Explain why your results do not contradict Fubini's Theorem.

40. If $f(x, y)$ is continuous over $R: a \le x \le b$, $c \le y \le d$ and

$$F(x, y) = \int_a^x\int_c^y f(u, v)\, dv\, du$$

on the interior of R, find the second partial derivatives F_{xy} and F_{yx}.

15.2 Double Integrals over General Regions

In this section we define and evaluate double integrals over bounded regions in the plane that are more general than rectangles. These double integrals are also evaluated as iterated integrals, with the main practical problem being that of determining the limits of integration. Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involve variables, not just constants.

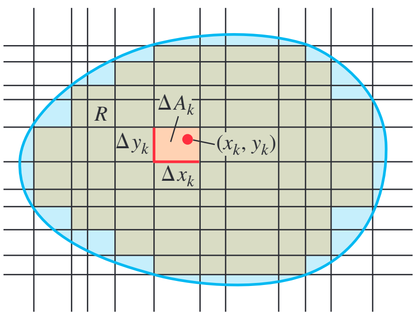


FIGURE 15.8 A rectangular grid partitioning a bounded, nonrectangular region into rectangular cells.

Double Integrals over Bounded, Nonrectangular Regions

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, such as the one in Figure 15.8, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R since its boundary is curved, and some of the small rectangles in the grid lie partly outside R. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

As the norm of the partition forming S_n goes to zero, $\|P\| \to 0$, the width and height of each enclosed rectangle go to zero, their area ΔA_k goes to zero, and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value that is not dependent on any of the choices we made. This limit is called the **double integral** of $f(x, y)$ over R:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA.$$

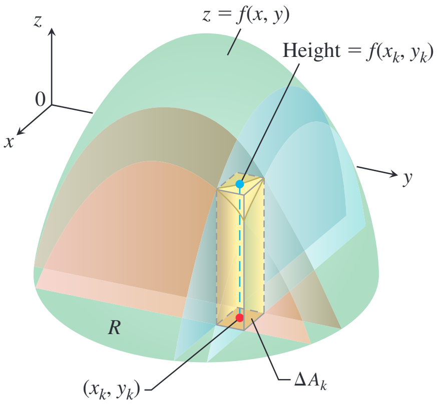


FIGURE 15.9 We define the volume of a solid with a curved base as a limit of the sums of volumes of approximating rectangular boxes.

The nature of the boundary of R introduces issues not found in integrals over an interval. When R has a curved boundary, the n rectangles of a partition lie inside R but do not cover all of R. In order for a partition to approximate R well, the parts of R covered by small rectangles lying partly outside R must become negligible as the norm of the partition approaches zero. This property of being nearly filled in by a partition of small norm is satisfied by all the regions that we will encounter. There is no problem with boundaries made from polygons, circles, and ellipses or from continuous graphs over an interval, joined end to end. A curve with a "fractal" type of shape would be problematic, but such curves arise rarely in most applications. A careful discussion of which types of regions R can be used for computing double integrals is left to a more advanced text.

Volumes

If $f(x, y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\,dA$, as before (Figure 15.9).

If R is a region like the one shown in the xy-plane in Figure 15.10, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\,dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx. \tag{1}$$

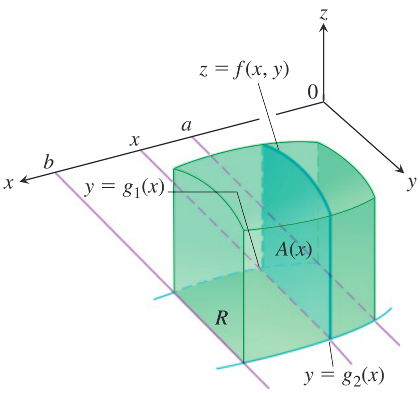


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

Similarly, if R is a region like the one shown in Figure 15.11, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy. \tag{2}$$

That the iterated integrals in Equations (1) and (2) both give the volume that we defined to be the double integral of f over R is a consequence of the following stronger form of Fubini's Theorem.

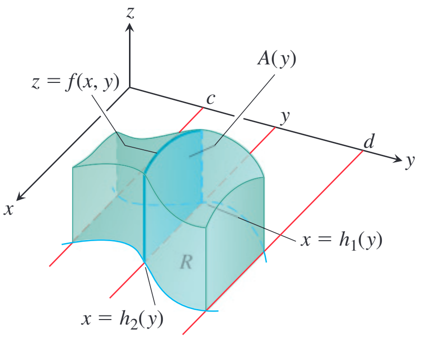


FIGURE 15.11 The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure 15.10 or in the way shown here. Both calculations have the same result.

THEOREM 2—Fubini's Theorem (Stronger Form)
Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

Some iterated double integrals we will encounter later in this text will use variables of integration other than x and y. For instance, we may write

$$\int_{u=p}^{u=q} \int_{v=G_1(u)}^{v=G_2(u)} F(u, v)\,dv\,du = \int_p^q \int_{G_1(u)}^{G_2(u)} F(u, v)\,dv\,du.$$

Regardless of which specific variables of integration are used, the limits of an iterated double integral always satisfy these properties:

- The limits of the *outside* integral are *constants* (they do not depend on either variable of integration), and
- the limits of the *inside* integral are *functions* that may depend on the variable of the *outside* integral.

EXAMPLE 1 Find the volume of the right prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12a. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be. ■

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other. The next example shows how this can happen.

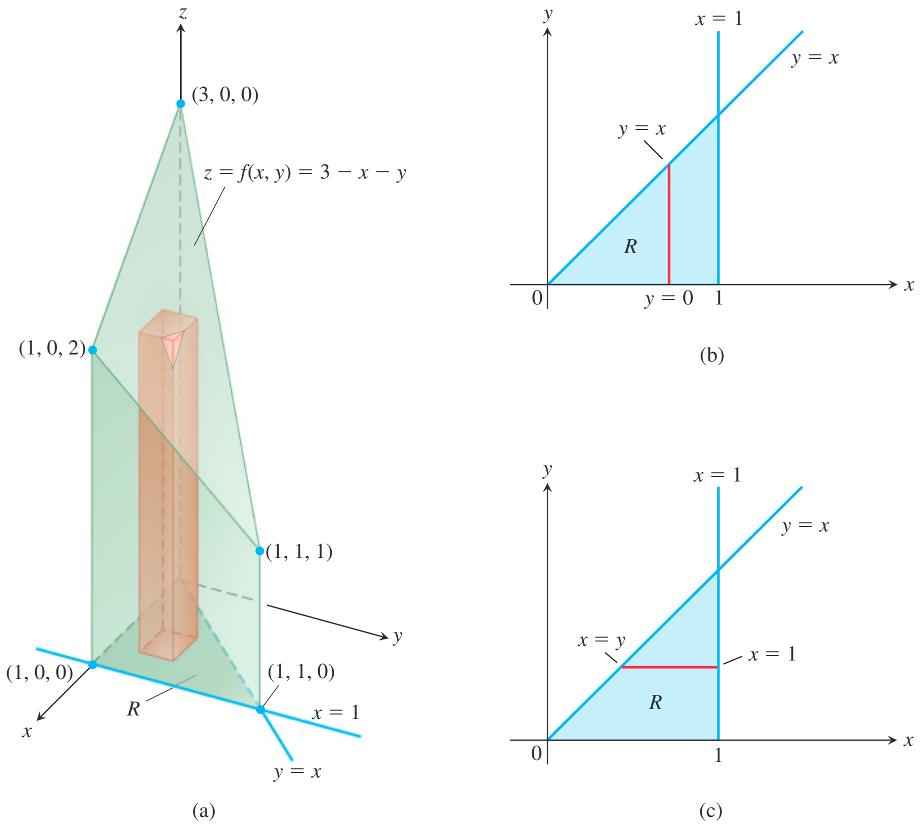


FIGURE 15.12 (a) Prism with a triangular base in the xy-plane. The volume of this prism is defined as a double integral over R. To evaluate it as an iterated integral, we may integrate first with respect to y and then with respect to x, or the other way around (Example 1). (b) Integration limits of

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x, y)\, dy\, dx.$$

If we integrate first with respect to y, we integrate along a vertical line through R and then integrate from left to right to include all the vertical lines in R. (c) Integration limits of

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x, y)\, dx\, dy.$$

If we integrate first with respect to x, we integrate along a horizontal line through R and then integrate from bottom to top to include all the horizontal lines in R.

EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

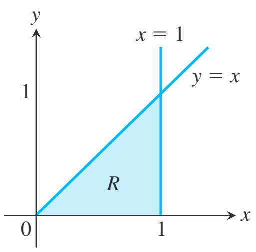


FIGURE 15.13 The region of integration in Example 2.

Solution The region of integration is shown in Figure 15.13. If we integrate first with respect to y and next with respect to x, then because x is held fixed in the first integration, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} dx = \int_0^1 \sin x\, dx = -\cos(1) + 1 \approx 0.46.$$

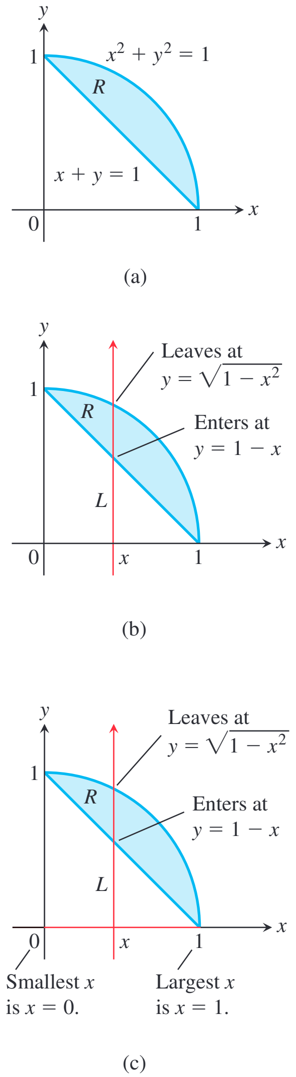


FIGURE 15.14 Finding the limits of integration when integrating first with respect to y and then with respect to x.

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy,$$

we run into a problem because $\int ((\sin x)/x)\, dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we may need to use numerical approximations. ■

Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-Sections When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves (Figure 15.14a).
2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure 15.14b).
3. *Find the x-limits of integration.* Choose x-limits that include all the vertical lines through R. These must be constants. The integral whose region of integration is shown in Figure 15.14c is

$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

Using Horizontal Cross-Sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3 (see Figure 15.15). The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

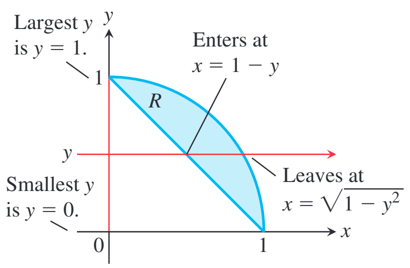


FIGURE 15.15 Finding the limits of integration when integrating first with respect to x and then with respect to y.

EXAMPLE 3 Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.

Solution The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 15.16a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure 15.16b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8. ■

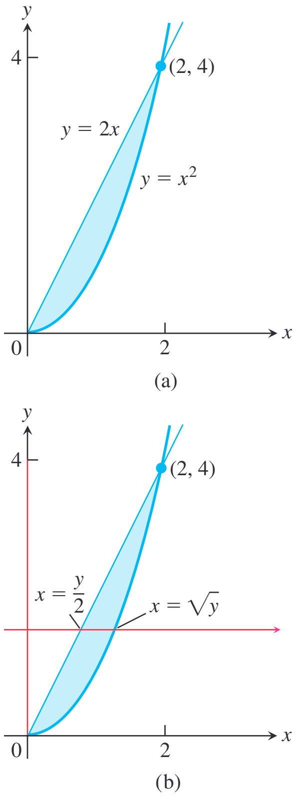


FIGURE 15.16 Region of integration for Example 3.

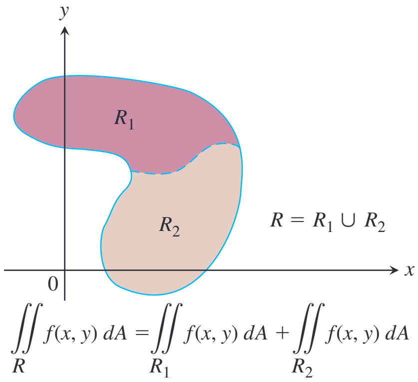


FIGURE 15.17 The Additivity Property for rectangular regions holds for regions bounded by smooth curves.

Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

> If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.
>
> **1.** *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA \quad (\text{any number } c)$
>
> **2.** *Sum and Difference:*
>
> $$\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA$$
>
> **3.** *Domination:*
>
> **(a)** $\displaystyle\iint_R f(x, y)\,dA \geq 0 \qquad \text{if} \qquad f(x, y) \geq 0 \text{ on } R$
>
> **(b)** $\displaystyle\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA \qquad \text{if} \qquad f(x, y) \geq g(x, y) \text{ on } R$
>
> **4.** *Additivity:* If R is the union of two nonoverlapping regions R_1 and R_2, then
>
> $$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$$

Property 4 assumes that the region of integration R is decomposed into nonoverlapping regions R_1 and R_2 with boundaries consisting of a finite number of line segments or smooth curves. Figure 15.17 illustrates an example of this property.

The idea behind these properties is that integrals behave like sums. If the function $f(x, y)$ is replaced by its constant multiple $cf(x, y)$, then a Riemann sum for f,

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k,$$

is replaced by a Riemann sum for cf:

$$\sum_{k=1}^{n} cf(x_k, y_k)\,\Delta A_k = c\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = cS_n.$$

Taking limits as $n \to \infty$ shows that $c\lim_{n\to\infty} S_n = c\iint_R f\,dA$ and $\lim_{n\to\infty} cS_n = \iint_R cf\,dA$ are equal. It follows that the Constant Multiple Property carries over from sums to double integrals.

The other properties are also easy to verify for Riemann sums, and carry over to double integrals for the same reason. While this discussion gives the idea, an actual proof that these properties hold requires a more careful analysis of how Riemann sums converge.

EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\,dx$ (first with respect to y and then with respect to x), two integrations

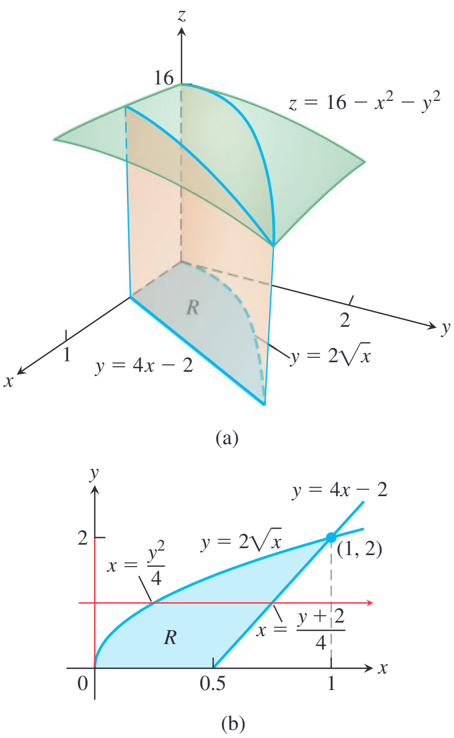


FIGURE 15.18 (a) The solid "wedge-like" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\,dy$.

will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

$$\begin{aligned}
\iint_R (16 - x^2 - y^2)\,dA
&= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\,dx\,dy \\
&= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx \\
&= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3\cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy \\
&= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4.
\end{aligned}$$

Our development of the double integral has focused on its representation of the volume of the solid region between R and the surface $z = f(x, y)$ of a positive continuous function. Just as we saw with signed area in the case of single integrals, when $f(x_k, y_k)$ is negative, the product $f(x_k, y_k)\Delta A_k$ is the negative of the volume of the rectangular box shown in Figure 15.9 that was used to form the approximating Riemann sum. So for an arbitrary continuous function f defined over R, the limit of any Riemann sum represents the *signed* volume (not the total volume) of the solid region between R and the surface. The double integral has other interpretations as well, and in the next section we will see how it is used to calculate the area of a general region in the plane.

EXERCISES 15.2

Sketching Regions of Integration

In Exercises 1–8, sketch the regions of integration associated with the given double integrals.

1. $\int_0^3 \int_0^{2x} f(x, y)\,dy\,dx$

2. $\int_{-1}^2 \int_{x-1}^{x^2} f(x, y)\,dy\,dx$

3. $\int_{-2}^2 \int_{y^2}^4 f(x, y)\,dx\,dy$

4. $\int_0^1 \int_y^{2y} f(x, y)\,dx\,dy$

5. $\int_0^1 \int_{e^x}^e f(x, y)\,dy\,dx$

6. $\int_1^{e^2} \int_0^{\ln x} f(x, y)\,dy\,dx$

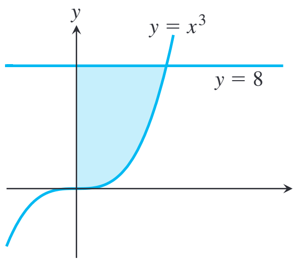
7. $\int_0^1 \int_0^{\arcsin y} f(x, y)\,dx\,dy$

8. $\int_0^8 \int_{y/4}^{y^{1/3}} f(x, y)\,dx\,dy$

Finding Limits of Integration

In Exercises 9–18, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

9.



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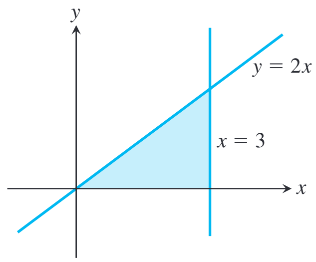

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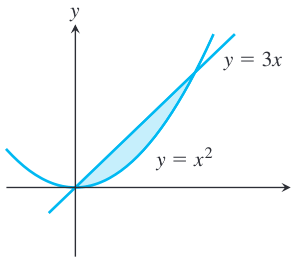

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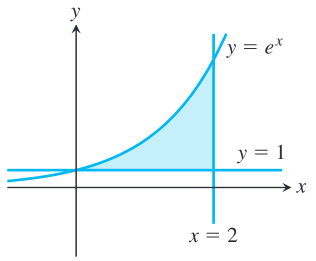

13. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$

14. Bounded by $y = \tan x$, $x = 0$, and $y = 1$

15. Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$

16. Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$

17. Bounded by $y = 3 - 2x$, $y = x$, and $x = 0$

18. Bounded by $y = x^2$ and $y = x + 2$

Evaluating Iterated Integrals

In Exercises 19–26, evaluate the integral.

19. $\int_1^2 \int_0^{2x} xy^3\, dy\, dx$

20. $\int_1^3 \int_y^{2y} y\, dx\, dy$

21. $\int_0^1 \int_y^1 (\sqrt{x} + xy)\, dx\, dy$

22. $\int_0^2 \int_0^{x^3} (y^2 - x)\, dy\, dx$

23. $\int_0^{\sqrt{\pi}} \int_0^{x^2} x \sin y\, dy\, dx$

24. $\int_0^1 \int_0^{\arctan y} \frac{1}{1 + y^2}\, dx\, dy$

25. $\int_1^4 \int_y^{y^2} \sqrt{\frac{y}{x}}\, dx\, dy$

26. $\int_3^5 \int_1^{e^x} \frac{1}{xy}\, dy\, dx$

Finding Regions of Integration and Double Integrals

In Exercises 27–32, sketch the region of integration and evaluate the integral.

27. $\int_0^\pi \int_0^x x \sin y\, dy\, dx$

28. $\int_0^\pi \int_0^{\sin x} y\, dy\, dx$

29. $\int_1^{\ln 8} \int_1^{\ln y} e^{x+y}\, dx\, dy$

30. $\int_1^2 \int_y^{y^2} dx\, dy$

31. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$

32. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx$

In Exercises 33–36, integrate f over the given region.

33. Quadrilateral $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, and $x = 2$

34. Triangle $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

35. Triangle $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$

36. Curved region $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 37–40 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

37. $\int_{-2}^0 \int_v^{-v} 2\, dp\, dv$ (the pv-plane)

38. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t\, dt\, ds$ (the st-plane)

39. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t\, du\, dt$ (the tu-plane)

40. $\int_0^{3/2} \int_1^{4-2u} \frac{4 - 2u}{v^2}\, dv\, du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 41–54, sketch the region of integration, and write an equivalent double integral with the order of integration reversed.

41. $\int_0^1 \int_2^{4-2x} dy\, dx$

42. $\int_0^2 \int_{y-2}^0 dx\, dy$

43. $\int_0^1 \int_y^{\sqrt{y}} dx\, dy$

44. $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

45. $\int_0^1 \int_1^{e^x} dy\, dx$

46. $\int_0^{\ln 2} \int_{e^y}^2 dx\, dy$

47. $\int_0^{3/2} \int_0^{9-4x^2} 16x\, dy\, dx$

48. $\int_0^2 \int_0^{4-y^2} y\, dx\, dy$

49. $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\, dx\, dy$

50. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\, dy\, dx$

51. $\int_1^e \int_0^{\ln x} xy\, dy\, dx$

52. $\int_0^{\pi/6} \int_{\sin x}^{1/2} xy^2\, dy\, dx$

53. $\int_0^3 \int_1^{e^y} (x + y)\, dx\, dy$

54. $\int_0^{\sqrt{3}} \int_0^{\tan^{-1} y} \sqrt{xy}\, dx\, dy$

In Exercises 55–64, sketch the region of integration, reverse the order of integration, and evaluate the integral.

55. $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$

56. $\int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx$

57. $\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$

58. $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y}\, dy\, dx$

59. $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx\, dy$

60. $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$

61. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\, dx\, dy$

62. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\, dx}{y^4 + 1}$

63. Square region $\iint_R (y - 2x^2)\, dA$ where R is the region bounded by the square $|x| + |y| = 1$

64. Triangular region $\iint_R xy\, dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

65. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

66. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

67. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

68. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

69. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

70. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

71. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

72. Find the volume of the solid cut from the square column $|x| + |y| \le 1$ by the planes $z = 0$ and $3x + z = 3$.

73. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.

74. Find the volume of the solid bounded on the front and back by the planes $x = \pm\pi/3$, on the sides by the cylinders $y = \pm\sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy-plane.

In Exercises 75 and 76, sketch the region of integration and the solid whose volume is given by the double integral.

75. $\displaystyle\int_0^3\int_0^{2-2x/3}\left(1 - \frac{1}{3}x - \frac{1}{2}y\right)dy\,dx$

76. $\displaystyle\int_0^4\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\sqrt{25 - x^2 - y^2}\,dx\,dy$

Integrals over Unbounded Regions

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits, as in Section 8.8. Evaluate the improper integrals in Exercises 77–80 as iterated integrals.

77. $\displaystyle\int_1^\infty\int_{e^{-x}}^1\frac{1}{x^3y}\,dy\,dx$

78. $\displaystyle\int_{-1}^1\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y + 1)\,dy\,dx$

79. $\displaystyle\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{(x^2 + 1)(y^2 + 1)}\,dx\,dy$

80. $\displaystyle\int_0^\infty\int_0^\infty xe^{-(x+2y)}\,dx\,dy$

Approximating Integrals with Finite Sums

In Exercises 81 and 82, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y)\,dA \approx \sum_{k=1}^n f(x_k, y_k)\,\Delta A_k$$

81. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

82. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the kth subrectangle (provided the subrectangle lies within R)

Theory and Examples

83. Circular sector Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \le 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.

84. Unbounded region Integrate $f(x, y) = 1/\left[(x^2 - x)(y - 1)^{2/3}\right]$ over the infinite rectangle $2 \le x < \infty$, $0 \le y \le 2$.

85. Noncircular cylinder A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1\int_0^y(x^2 + y^2)\,dx\,dy + \int_1^2\int_0^{2-y}(x^2 + y^2)\,dx\,dy.$$

Sketch the base region R, and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

86. Converting to a double integral Evaluate the integral

$$\int_0^2(\arctan \pi x - \arctan x)\,dx.$$

(*Hint:* Write the integrand as an integral.)

87. Maximizing a double integral What region R in the xy-plane maximizes the value of

$$\iint_R(4 - x^2 - 2y^2)\,dA?$$

Give reasons for your answer.

88. Minimizing a double integral What region R in the xy-plane minimizes the value of

$$\iint_R(x^2 + y^2 - 9)\,dA?$$

Give reasons for your answer.

89. Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

90. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices $(0, 1)$, $(2, 0)$, and $(1, 2)$? Give reasons for your answer.

91. Unbounded region Prove that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,dx\,dy = \lim_{b\to\infty}\int_{-b}^b\int_{-b}^b e^{-x^2-y^2}\,dx\,dy = 4\left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

92. Improper double integral Evaluate the improper integral

$$\int_0^1\int_0^3\frac{x^2}{(y - 1)^{2/3}}\,dy\,dx.$$

COMPUTER EXPLORATIONS

Use a CAS double-integral evaluator to estimate the values of the integrals in Exercises 93–96.

93. $\displaystyle\int_1^3\int_1^x\frac{1}{xy}\,dy\,dx$

94. $\displaystyle\int_0^1\int_0^1 e^{-(x^2+y^2)}\,dy\,dx$

95. $\displaystyle\int_0^1\int_0^1\arctan xy\,dy\,dx$

96. $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}}3\sqrt{1 - x^2 - y^2}\,dy\,dx$

Use a CAS double-integral evaluator to find the integrals in Exercises 97–102. Then reverse the order of integration and evaluate, again with a CAS.

97. $\int_0^1 \int_{2y}^4 e^{x^2}\, dx\, dy$

98. $\int_0^3 \int_{x^2}^9 x \cos(y^2)\, dy\, dx$

99. $\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\, dx\, dy$

100. $\int_0^2 \int_0^{4-y^2} e^{xy}\, dx\, dy$

101. $\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx$

102. $\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$

15.3 Area by Double Integration

In this section we show how to use double integrals to calculate the areas of bounded regions in the plane, and to find the average value of a function of two variables.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k. \qquad (1)$$

This is simply the sum of the areas of the small rectangles in the partition of R, and it approximates what we would like to call the area of R. As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete (Figure 15.8). We define the area of R to be the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA. \qquad (2)$$

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R.

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure 15.19), noting where the two curves intersect at the origin and $(1, 1)$, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Notice that the single-variable integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.6. ■

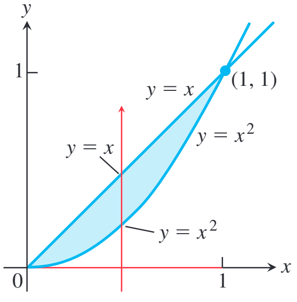

FIGURE 15.19 The region in Example 1.