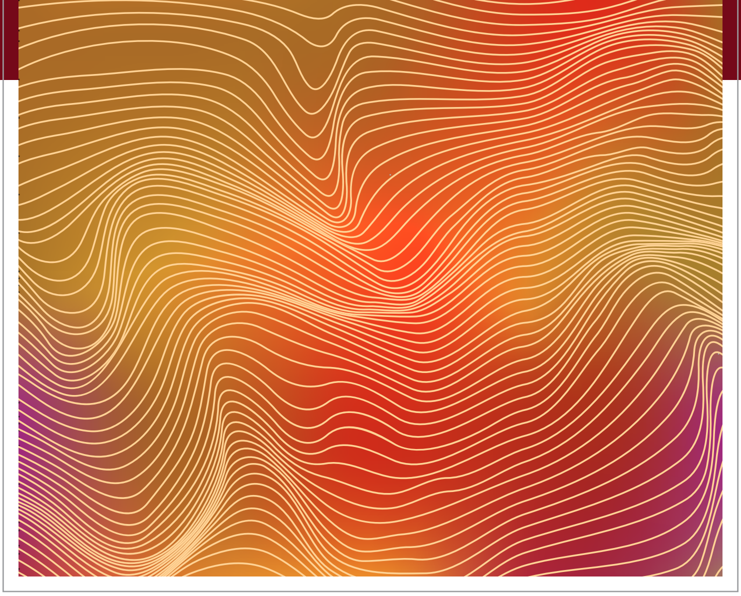
15

Multiple Integrals



OVERVIEW In this chapter we define the *double integral* of a function of two variables $f(x, y)$ over a region in the plane as the limit of approximating Riemann sums. Just as a single integral can represent signed area, so can a double integral represent signed volume. Double integrals can be evaluated using the Fundamental Theorem of Calculus studied in Section 5.4, but now the evaluations are done twice by integrating with respect to each of the variables x and y in turn. Double integrals can be used to find areas of more general regions in the plane than those encountered in Chapter 5. Moreover, just as the Substitution Rule could simplify finding single integrals, we can sometimes use polar coordinates to simplify computing a double integral. We study more general substitutions for evaluating double integrals as well.

We also define the *triple integral* of a function of three variables $f(x, y, z)$ over a region in space. Triple integrals can be used to find volumes of still more general regions in space, and their evaluation is like that of double integrals with yet a third evaluation. *Cylindrical* or *spherical coordinates* can sometimes be used to simplify the calculation of a triple integral, and we investigate those techniques. Double and triple integrals have a number of applications, such as calculating the average value of a multivariable function, and finding moments and centers of mass.

15.1 Double and Iterated Integrals over Rectangles

In Chapter 5 we defined the definite integral of a function $f(x)$ over an interval $[a, b]$ as a limit of Riemann sums. In this section we extend this idea to define the *double integral* of a function of two variables $f(x, y)$ over a bounded rectangle R in the plane. The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of f at a point c_k inside that subinterval, and then adding together all the products. A similar method of partitioning, multiplying, and summing is used to construct double integrals as limits of approximating Riemann sums.

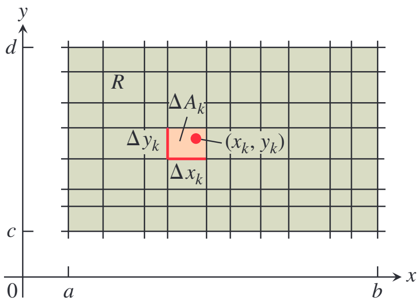


FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \, \Delta y_k$.

Double Integrals

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region R,

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

We subdivide R into small rectangles using a network of lines parallel to the x- and y-axes (Figure 15.1). The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a **partition** of R. A small rectangular piece of width Δx and height Δy has area

$\Delta A = \Delta x \, \Delta y$. If we number the small pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where ΔA_k is the area of the kth small rectangle.

To form a Riemann sum over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at that point by the area ΔA_k, and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

Depending on how we pick (x_k, y_k) in the kth small rectangle, we may get different values for S_n.

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero. The **norm** of a partition P, written $\|P\|$, is the largest width or height of any rectangle in the partition. If $\|P\| = 0.1$, then all the rectangles in the partition of R have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of P goes to zero, which is written $\|P\| \to 0$. The resulting limit is then written as

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

As $\|P\| \to 0$ and the rectangles get narrow and short, their number n increases, so we can also write this limit as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k,$$

with the understanding that $\|P\| \to 0$, and hence $\Delta A_k \to 0$, as $n \to \infty$.

Many choices are involved in a limit of this kind. The collection of small rectangles is determined by the grid of vertical and horizontal lines that determine a rectangular partition of R. In each of the resulting small rectangles there is a choice of an arbitrary point (x_k, y_k) at which f is evaluated. These choices together determine a single Riemann sum. To form a limit, we repeat the whole process again and again, choosing partitions whose rectangle widths and heights both go to zero and whose number goes to infinity.

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R, which is written as

$$\iint_R f(x, y)\,dA \qquad \text{or} \qquad \iint_R f(x, y)\,dx\,dy.$$

It can be shown that if $f(x, y)$ is a continuous function throughout R, then f is integrable, as in the single-variable case discussed in Chapter 5. Many discontinuous functions are also integrable, including functions that are discontinuous only on a finite number of points or smooth curves. We leave the proof of these facts to a more advanced text.

Double Integrals as Volumes

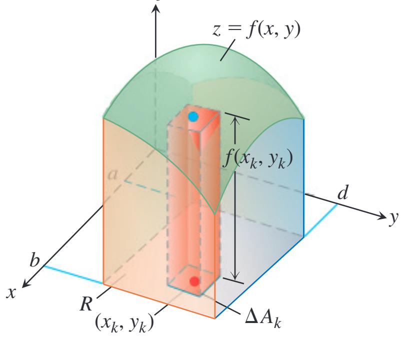


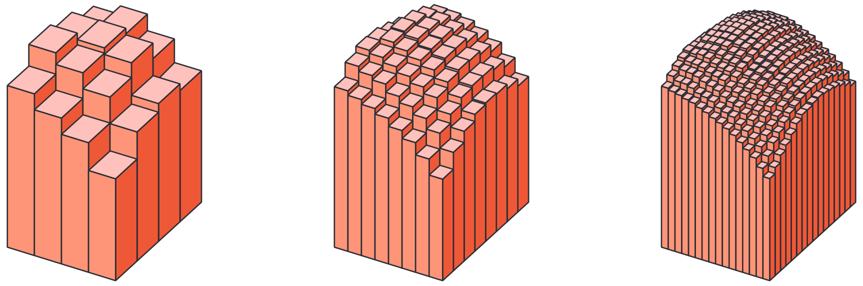
FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.

When $f(x, y)$ is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the three-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$ (Figure 15.2). Each term $f(x_k, y_k)\,\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\,\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid. We *define* this volume to be

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)\,dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.

As you might expect, this more general method of calculating volume agrees with the methods in Chapter 6, but we do not prove this here. Figure 15.3 shows Riemann sum approximations to the volume becoming more accurate as the number n of boxes increases.



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 15.2.

Fubini's Theorem for Calculating Double Integrals

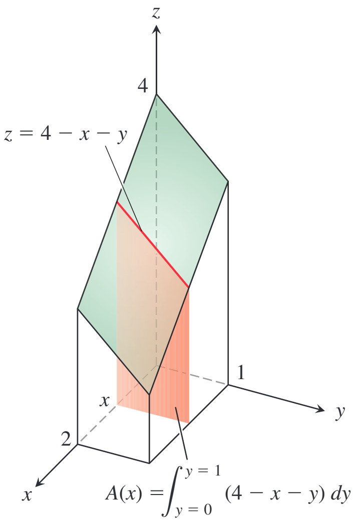


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: 0 \leq x \leq 2,\ 0 \leq y \leq 1$ in the xy-plane. If we apply the method of slicing from Section 6.1, with slices perpendicular to the x-axis (Figure 15.4), then the volume is

$$\int_{x=0}^{x=2} A(x)\,dx, \tag{1}$$

where $A(x)$ is the cross-sectional area at x. For each value of x, we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\,dy, \tag{2}$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining Equations (1) and (2), we see that the volume of the entire solid is

$$\begin{aligned}
\text{Volume} &= \int_{x=0}^{x=2} A(x)\,dx = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=1}(4 - x - y)\,dy\right)dx \\
&= \int_{x=0}^{x=2}\left[4y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2}\left(\frac{7}{2} - x\right)dx \\
&= \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = 5.
\end{aligned}$$

We often omit parentheses separating the two integrals in the formula above and write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx. \tag{3}$$

The expression on the right, called an **iterated** or **repeated integral**, says that the volume is obtained by integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$ while holding x fixed, and then integrating the resulting expression in x from $x = 0$ to $x = 2$. The limits of integration 0 and 1 are associated with y, so they are placed on the integral closest to dy. The other limits of integration, 0 and 2, are associated with the variable x, so they are placed on the outside integral symbol that is paired with dx.

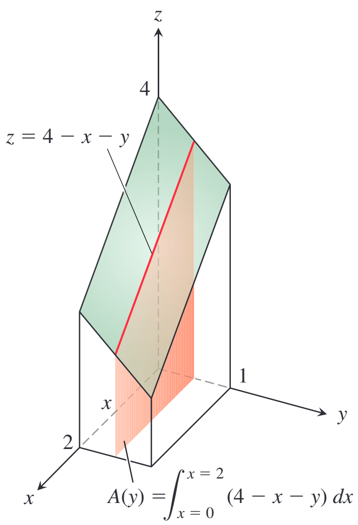


FIGURE 15.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x.

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis (Figure 15.5)? As a function of y, the typical cross-sectional area is

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\, dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y. \tag{4}$$

The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\, dy = \int_{y=0}^{y=1} (6 - 2y)\, dy = \left[6y - y^2\right]_0^1 = 5,$$

in agreement with our earlier calculation.

Again, we may give a formula for the volume as an iterated integral by writing

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy.$$

The expression on the right says we can find the volume by integrating $4 - x - y$ with respect to x from $x = 0$ to $x = 2$ as in Equation (4) and integrating the result with respect to y from $y = 0$ to $y = 1$. In this iterated integral, the order of integration is first x and then y, the reverse of the order in Equation (3).

What do these two volume calculations with iterated integrals have to do with the double integral

$$\iint_R (4 - x - y)\, dA$$

over the rectangle $R: 0 \le x \le 2,\ 0 \le y \le 1$? The answer is that both iterated integrals give the value of the double integral. This is what we would reasonably expect, since the double integral measures the volume of the same region as the two iterated integrals. A theorem published in 1907 by Guido Fubini says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration. (Fubini proved his theorem in greater generality, but this is what it says in our setting.)

HISTORICAL BIOGRAPHY
Guido Fubini
(1879–1943)
www.bit.ly/2xOIk22

THEOREM 1—Fubini's Theorem (First Form)
If $f(x, y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time using the Fundamental Theorem of Calculus.

Fubini's Theorem also says that we may calculate the double integral by integrating in *either* order, a genuine convenience. When we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to the y-axis.

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R:\ 0 \le x \le 2,\ -1 \le y \le 1.$$

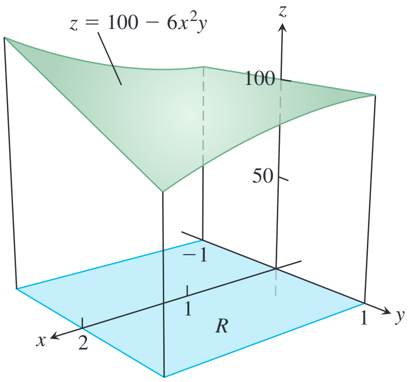


FIGURE 15.6 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 1).

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\,dA = \int_{-1}^{1}\int_{0}^{2}(100 - 6x^2y)\,dx\,dy = \int_{-1}^{1}\Big[100x - 2x^3y\Big]_{x=0}^{x=2}dy$$

$$= \int_{-1}^{1}(200 - 16y)\,dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1}(100 - 6x^2y)\,dy\,dx = \int_0^2\Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1}dx$$

$$= \int_0^2[(100 - 3x^2) - (-100 - 3x^2)]\,dx$$

$$= \int_0^2 200\,dx = 400.$$

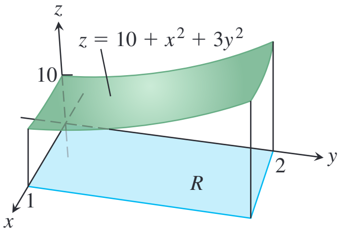


FIGURE 15.7 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 2).

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \le x \le 1,\ 0 \le y \le 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\,dA = \int_0^1\int_0^2(10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1\Big[10y + x^2y + y^3\Big]_{y=0}^{y=2}dx$$

$$= \int_0^1(28 + 2x^2)\,dx = \Big[28x + \frac{2}{3}x^3\Big]_0^1 = \frac{86}{3}.$$

EXERCISES 15.1

Evaluating Iterated Integrals

In Exercises 1–14, evaluate the iterated integral.

1. $\int_1^2\int_0^4 2xy\,dy\,dx$
2. $\int_0^2\int_{-1}^{1}(x - y)\,dy\,dx$
3. $\int_{-1}^{0}\int_{-1}^{1}(x + y + 1)\,dx\,dy$
4. $\int_0^1\int_0^1\left(1 - \frac{x^2 + y^2}{2}\right)dx\,dy$
5. $\int_0^3\int_0^2(4 - y^2)\,dy\,dx$
6. $\int_0^3\int_{-2}^{0}(x^2y - 2xy)\,dy\,dx$
7. $\int_0^1\int_0^1\frac{y}{1 + xy}\,dx\,dy$
8. $\int_1^4\int_0^4\left(\frac{x}{2} + \sqrt{y}\right)dx\,dy$
9. $\int_0^{\ln 2}\int_1^{\ln 5}e^{2x+y}\,dy\,dx$
10. $\int_0^1\int_1^2 xye^x\,dy\,dx$
11. $\int_{-1}^{2}\int_0^{\pi/2} y\sin x\,dx\,dy$
12. $\int_{\pi}^{2\pi}\int_0^{\pi}(\sin x + \cos y)\,dx\,dy$
13. $\int_1^4\int_1^e\frac{\ln x}{xy}\,dx\,dy$
14. $\int_{-1}^{2}\int_1^2 x\ln y\,dy\,dx$
15. Find all values of the constant c so that $\int_0^1\int_0^c(2x + y)\,dx\,dy = 3.$
16. Find all values of the constant c so that
$$\int_{-1}^{c}\int_0^2(xy + 1)\,dy\,dx = 4 + 4c.$$

Evaluating Double Integrals over Rectangles

In Exercises 17–24, evaluate the double integral over the given region R.

17. $\iint_R (6y^2 - 2x)\,dA, \quad R:\ 0 \le x \le 1,\ 0 \le y \le 2$
18. $\iint_R \left(\frac{\sqrt{x}}{y^2}\right)dA, \quad R:\ 0 \le x \le 4,\ 1 \le y \le 2$

19. $\iint_R xy \cos y \, dA, \quad R: \ -1 \le x \le 1, \ \ 0 \le y \le \pi$

20. $\iint_R y \sin(x + y) \, dA, \quad R: \ -\pi \le x \le 0, \ \ 0 \le y \le \pi$

21. $\iint_R e^{x-y} \, dA, \quad R: \ 0 \le x \le \ln 2, \ \ 0 \le y \le \ln 2$

22. $\iint_R xye^{xy^2} \, dA, \quad R: \ 0 \le x \le 2, \ \ 0 \le y \le 1$

23. $\iint_R \dfrac{xy^3}{x^2 + 1} \, dA, \quad R: \ 0 \le x \le 1, \ \ 0 \le y \le 2$

24. $\iint_R \dfrac{y}{x^2y^2 + 1} \, dA, \quad R: \ 0 \le x \le 1, \ \ 0 \le y \le 1$

In Exercises 25 and 26, integrate f over the given region.

25. Square $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

26. Rectangle $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

In Exercises 27 and 28, sketch the solid whose volume is given by the specified integral.

27. $\int_0^1 \int_0^2 (9 - x^2 - y^2) \, dy \, dx$ **28.** $\int_0^3 \int_1^4 (7 - x - y) \, dx \, dy$

29. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

30. Find the volume of the region bounded above by the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: \ 0 \le x \le 2$, $0 \le y \le 2$.

31. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1$, $0 \le y \le 1$.

32. Find the volume of the region bounded above by the plane $z = y/2$ and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$.

33. Find the volume of the region bounded above by the surface $z = 2 \sin x \cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

34. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$, $0 \le y \le 2$.

35. Find a value of the constant k so that $\int_1^2 \int_0^3 kx^2 y \, dx \, dy = 1$.

36. Evaluate $\int_{-1}^1 \int_0^{\pi/2} x \sin \sqrt{y} \, dy \, dx$.

37. Use Fubini's Theorem to evaluate

$$\int_0^2 \int_0^1 \frac{x}{1 + xy} \, dx \, dy.$$

38. Use Fubini's Theorem to evaluate

$$\int_0^1 \int_0^3 xe^{xy} \, dx \, dy.$$

T **39.** Use a software application to compute the integrals

a. $\int_0^1 \int_0^2 \dfrac{y - x}{(x + y)^3} \, dx \, dy$ **b.** $\int_0^2 \int_0^1 \dfrac{y - x}{(x + y)^3} \, dy \, dx$

Explain why your results do not contradict Fubini's Theorem.

40. If $f(x, y)$ is continuous over $R: a \le x \le b, c \le y \le d$ and

$$F(x, y) = \int_a^x \int_c^y f(u, v) \, dv \, du$$

on the interior of R, find the second partial derivatives F_{xy} and F_{yx}.

15.2 Double Integrals over General Regions

In this section we define and evaluate double integrals over bounded regions in the plane that are more general than rectangles. These double integrals are also evaluated as iterated integrals, with the main practical problem being that of determining the limits of integration. Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involve variables, not just constants.

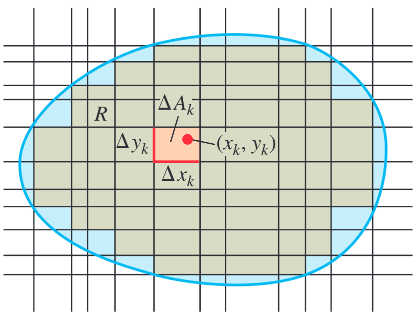


FIGURE 15.8 A rectangular grid partitioning a bounded, nonrectangular region into rectangular cells.

Double Integrals over Bounded, Nonrectangular Regions

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, such as the one in Figure 15.8, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R since its boundary is curved, and some of the small rectangles in the grid lie partly outside R. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.