14.5 Directional Derivatives and Gradient Vectors

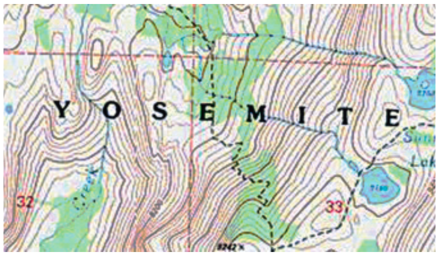

FIGURE 14.26 Contours within Yosemite National Park in California show streams, which follow paths of steepest descent, running perpendicular to the contours. (*Source:* Yosemite National Park Map from U.S. Geological Survey, http://www.usgs.gov)

If you look at the map (Figure 14.26) showing contours within Yosemite National Park in California, you will notice that the streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach lower elevations as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above sea level has a particular direction. In this section, you will see why this direction, called the "downhill" direction, is perpendicular to the contours.

Directional Derivatives in the Plane

We know from Section 14.4 that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t)$, $y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of motion along the curve. If the curve is a straight line and t is the arc length parameter along the line measured from P_0 in the direction of a given unit vector $\mathbf{u}$, then df/dt is the rate of change of f with respect to distance in its domain in the direction of $\mathbf{u}$. By varying $\mathbf{u}$, we find the rates at which f changes with respect to distance as we move through P_0 in different directions. We now define this idea more precisely.

Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. Then the equations

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

parametrize the line through P_0 parallel to $\mathbf{u}$. If the parameter s measures arc length from P_0 in the direction of $\mathbf{u}$, we find the rate of change of f at P_0 in the direction of $\mathbf{u}$ by calculating df/ds at P_0 (Figure 14.27).

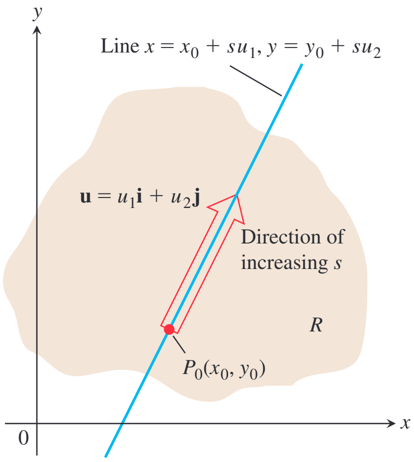

FIGURE 14.27 The rate of change of f in the direction of $\mathbf{u}$ at a point P_0 is the rate at which f changes along this line at P_0.

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$$D_{\mathbf{u}}f(P_0) \qquad \text{or} \qquad D_{\mathbf{u}}f|_{P_0}.$$

"The derivative of f in the direction of $\mathbf{u}$, evaluated at P_0"

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of f at P_0 in the $\mathbf{i}$ and $\mathbf{j}$ directions. This observation can be seen by comparing Equation (1) to the definitions of the two partial derivatives given in Section 14.3.

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$.

Solution Applying the definition in Equation (1), we obtain

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s} \qquad \text{Eq. (1)}$$

$$= \lim_{s\to 0}\frac{f\left(1+s\cdot\frac{1}{\sqrt{2}}, 2+s\cdot\frac{1}{\sqrt{2}}\right) - f(1,2)}{s} \qquad \text{Substitute.}$$

$$= \lim_{s\to 0}\frac{\left(1+\frac{s}{\sqrt{2}}\right)^2 + \left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right) - (1^2+1\cdot 2)}{s}$$

$$= \lim_{s\to 0}\frac{\left(1+\frac{2s}{\sqrt{2}}+\frac{s^2}{2}\right) + \left(2+\frac{3s}{\sqrt{2}}+\frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s\to 0}\frac{\frac{5s}{\sqrt{2}}+s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}}+s\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$.

Interpretation of the Directional Derivative

The equation $z = f(x,y)$ represents a surface S in space. If $z_0 = f(x_0,y_0)$, then the point $P(x_0,y_0,z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0,y_0)$ parallel to $\mathbf{u}$ intersects S in a curve C (Figure 14.28). The rate of change of f in the direction of $\mathbf{u}$ is the slope of the tangent to C at P in the right-handed system formed by the vectors $\mathbf{u}$ and $\mathbf{k}$.

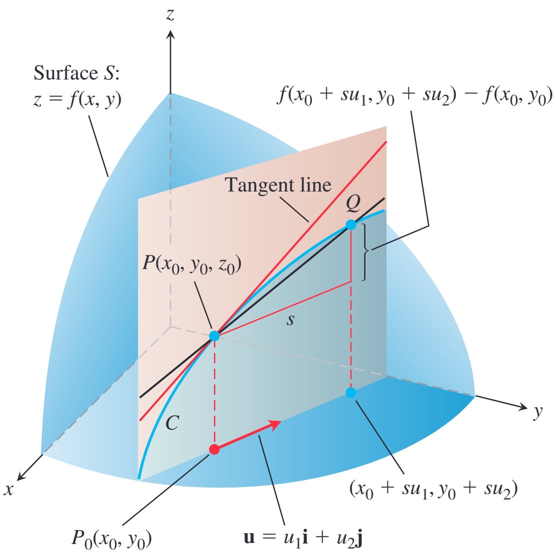

FIGURE 14.28 The slope of the trace curve C at P_0 is $\lim_{Q\to P}$ slope (PQ); this is the directional derivative

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = D_{\mathbf{u}}f\Big|_{P_0}.$$

When $\mathbf{u} = \mathbf{i}$, the directional derivative at P_0 is $\partial f/\partial x$ evaluated at (x_0,y_0). When $\mathbf{u} = \mathbf{j}$, the directional derivative at P_0 is $\partial f/\partial y$ evaluated at (x_0,y_0). The directional derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction $\mathbf{u}$, not just in the directions $\mathbf{i}$ and $\mathbf{j}$.

For a physical interpretation of the directional derivative, suppose that $T = f(x,y)$ is the temperature at each point (x,y) over a region in the plane. Then $f(x_0,y_0)$ is the temperature at the point $P_0(x_0, y_0)$, and $D_{\mathbf{u}}f|_{P_0}$ is the instantaneous rate of change of the temperature at P_0 stepping off in the direction $\mathbf{u}$.

Calculation and Gradients

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line

$$x = x_0 + su_1, \quad y = y_0 + su_2, \tag{2}$$

through $P_0(x_0, y_0)$, parametrized with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then, by the Chain Rule we find

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \left.\frac{\partial f}{\partial x}\right|_{P_0} \frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{P_0} \frac{dy}{ds} && \text{Chain Rule for differentiable } f \\
&= \left.\frac{\partial f}{\partial x}\right|_{P_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{P_0} u_2 && \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2 \\
&= \underbrace{\left[\left.\frac{\partial f}{\partial x}\right|_{P_0} \mathbf{i} + \left.\frac{\partial f}{\partial y}\right|_{P_0} \mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}.
\end{aligned} \tag{3}$$

Equation (3) says that the derivative of a differentiable function f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with a special vector, which we now define.

DEFINITION The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f|_{P_0} \qquad \text{or} \qquad \nabla f(x_0, y_0).$$

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f. Using the gradient notation, we restate Equation (3) as a theorem.

THEOREM 9—The Directional Derivative Is a Dot Product
If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \nabla f|_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 with the vector $\mathbf{u}$. In brief, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

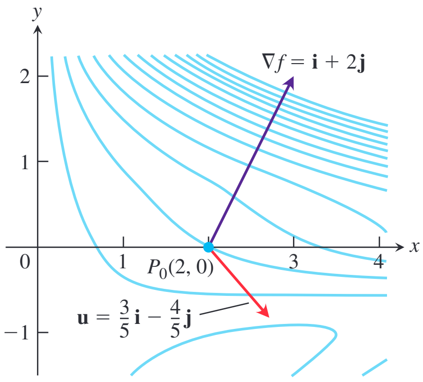

FIGURE 14.29 Picture ∇f as a vector in the domain of f. The figure shows a number of level curves of f. The rate at which f changes at $(2, 0)$ in the direction $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = -1$, which is the component of ∇f in the direction of unit vector $\mathbf{u}$ (Example 2).

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))\Big|_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))\Big|_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.29). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4) with the } D_{\mathbf{u}}f|_{P_0} \text{ notation}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

Evaluating the dot product in the brief version of Equation (4) gives

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f, and reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f|\cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

As we discuss later, these properties hold in three dimensions as well as two.

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point $(1, 1)$, and

(b) decreases most rapidly at $(1, 1)$.

(c) What are the directions of zero change in f at $(1, 1)$?

Solution

(a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$\nabla f|_{(1,1)} = (x\mathbf{i} + y\mathbf{j})\Big|_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

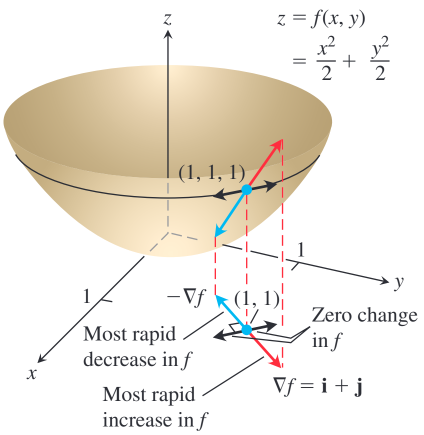

FIGURE 14.30 The direction in which $f(x, y)$ increases most rapidly at $(1, 1)$ is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at $(1, 1, 1)$ (Example 3).

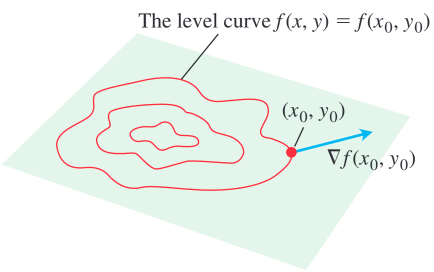

FIGURE 14.31 When it is nonzero, the gradient of a differentiable function of two variables at a point is always normal to the function's level curve through that point.

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.30. ■

Gradients and Tangents to Level Curves

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve part of a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule} \qquad (5)$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0.$$

Assuming the gradient of f is a nonzero vector, Equation (5) says that ∇f is normal to the tangent vector $d\mathbf{r}/dt$, so it is normal to the curve. This is seen in Figure 14.31.

> At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$ where the gradient of f is a nonzero vector, this vector is normal to the level curve through (x_0, y_0) (Figure 14.31).

Equation (5) validates our observation that streams flow perpendicular to the contours in topographical maps (see Figure 14.26). Since the downflowing stream will reach its destination in the fastest way, it must flow in the direction of the negative gradient vectors from Property 2 for the directional derivative. Equation (5) tells us these directions are perpendicular to the level curves.

This observation also enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients. The line through a point $P_0(x_0, y_0)$ normal to a nonzero vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0$$

(Exercise 39). If $\mathbf{N}$ is the gradient $\nabla f|_{(x_0, y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$, and this gradient is not the zero vector, then this equation gives the following formula.

> **Equation for the Tangent Line to a Level Curve**
>
> $$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \qquad (6)$$

EXAMPLE 4 Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

(Figure 14.32) at the point $(-2, 1)$.

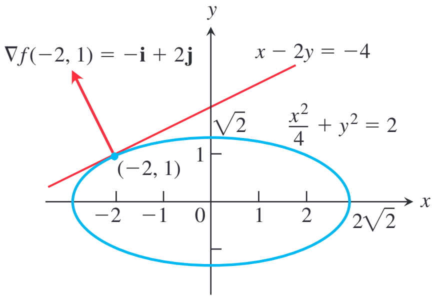

FIGURE 14.32 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)\Big|_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

Because this gradient vector is nonzero, the tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0 \qquad \text{Eq. (6)}$$
$$x - 2y = -4. \qquad \text{Simplify.}$$

If we know the gradients of two functions f and g, we automatically know the gradients of their sum, difference, constant multiples, product, and quotient. You are asked to establish the following rules in Exercise 40. Notice that these rules have the same form as the corresponding rules for derivatives of single-variable functions.

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

and

$$f\nabla g + g\nabla f = (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j}) \qquad \text{Substitute.}$$
$$= 3y\mathbf{i} + (3x - 6y)\mathbf{j}. \qquad \text{Simplify.}$$

We have therefore verified that for this example, $\nabla(fg) = f\nabla g + g\nabla f$.

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

so the properties listed earlier for functions of two variables extend to three variables. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f|_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3. \quad ■$$

Functions of More Than Three Variables

The gradient of a differentiable function of n variables $f(x_1, x_2, \ldots, x_n)$ is

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right\rangle.$$

If $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ is an n-dimensional vector such that $u_1^2 + u_2^2 + \cdots + u_n^2 = 1$ (so $\mathbf{u}$ is a unit vector since $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = 1$), then the directional derivative of f in the direction of $\mathbf{u}$ is

$$D_u f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x_1}u_1 + \frac{\partial f}{\partial x_2}u_2 + \cdots + \frac{\partial f}{\partial x_n}u_n.$$

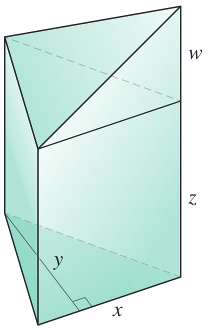

FIGURE 14.33 A tetrahedron on top of a triangular prism (Example 7).

EXAMPLE 7 The volume of the solid shown in Figure 14.33 consisting of a tetrahedron on top of a triangular prism is given by $f(x, y, z, w) = \frac{xy}{2}\left(z + \frac{w}{3}\right)$.

(a) Calculate the derivative of $f(x, y, z, w)$ at the point $P_0(6, 5, 8, 4)$ in the direction of $\mathbf{v} = \langle 1, -1, -1, 1\rangle$.

(b) What is the geometric significance of the value obtained in part (a)?

Solution

(a) The direction of $\mathbf{v}$ is the unit vector

$$\begin{aligned}\mathbf{u} &= \frac{1}{|\mathbf{v}|}\mathbf{v}\\ &= \frac{1}{\sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2}}\langle 1, -1, -1, 1\rangle\\ &= \frac{1}{2}\langle 1, -1, -1, 1\rangle\\ &= \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\rangle.\end{aligned}$$

The four partial derivatives of f at the point P_0 are

$$\begin{aligned}f_x &= \frac{y}{2}\left(z + \frac{w}{3}\right)\bigg|_{(6,5,8,4)} = \frac{70}{3},\\ f_y &= \frac{x}{2}\left(z + \frac{w}{3}\right)\bigg|_{(6,5,8,4)} = 28,\\ f_z &= \frac{xy}{2}\bigg|_{(6,5,8,4)} = 15,\\ f_w &= \frac{xy}{6}\bigg|_{(6,5,8,4)} = 5.\end{aligned}$$

The gradient of f at P_0 is

$$\nabla f|_{(6,5,8,4)} = \left\langle \frac{70}{3}, 28, 15, 5\right\rangle.$$

The derivative of f at $(6, 5, 8, 4)$ in the direction of $\mathbf{v}$ is

$$\begin{aligned}D_{\mathbf{u}}f|_{P_0} &= \nabla f|_{P_0} \cdot \mathbf{u}\\ &= \left\langle \frac{70}{3}, 28, 15, 5\right\rangle \cdot \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\rangle\\ &= \left(\frac{70}{3}\right)\left(\frac{1}{2}\right) + (28)\left(-\frac{1}{2}\right) + (15)\left(-\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right)\\ &= -\frac{22}{3}.\end{aligned}$$

(b) Geometrically, this means that if the dimensions are $x = 6$, $y = 5$, $z = 8$, and $w = 4$, and the dimensions are changed by moving at unit speed so that x and w increase at the same rate while both y and z decrease at that rate, then the volume of the solid decreases at the rate of 7 1/3.

The Chain Rule for Paths

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth path C, and $w = f(\mathbf{r}(t))$ is a scalar function evaluated along C, then according to the Chain Rule, Theorem 6 in Section 14.4,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

The partial derivatives on the right-hand side of the above equation are evaluated along the curve $\mathbf{r}(t)$, and the derivatives of the intermediate variables are evaluated at t. If we express this equation using vector notation, we have

The Derivative Along a Path

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t). \tag{7}$$

What Equation (7) says is that the derivative of the composite function $f(\mathbf{r}(t))$ is the "derivative" (gradient) of the outside function f, evaluated at $\mathbf{r}(t)$, "times" (dot product) the derivative of the inside function $\mathbf{r}$. This is analogous to the "Outside-Inside" Rule for derivatives of composite functions studied in Section 3.6. That is, the multivariable Chain Rule for paths has exactly *the same form* as the rule for single-variable differential calculus when appropriate interpretations are given to the meanings of the terms and operations involved.

EXERCISES 14.5

Calculating Gradients

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient, together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$
2. $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$
3. $g(x, y) = xy^2, \quad (2, -1)$
4. $g(x, y) = \dfrac{x^2}{2} - \dfrac{y^2}{2}, \quad (\sqrt{2}, 1)$
5. $f(x, y) = \sqrt{2x + 3y}, \quad (-1, 2)$
6. $f(x, y) = \tan^{-1} \dfrac{\sqrt{x}}{y}, \quad (4, -2)$

In Exercises 7–10, find ∇f at the given point.

7. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, \quad (1, 1, 1)$
8. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \arctan xz, \quad (1, 1, 1)$
9. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$
10. $f(x, y, z) = e^{x+y} \cos z + (y + 1) \arcsin x, \quad (0, 0, \pi/6)$

Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at P_0 in the direction of $\mathbf{v}$.

11. $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$
12. $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$
13. $g(x, y) = \dfrac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{v} = 12\mathbf{i} + 5\mathbf{j}$
14. $h(x, y) = \arctan(y/x) + \sqrt{3} \arcsin(xy/2), \quad P_0(1, 1),$ $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$
15. $f(x, y, z) = xy + yz + zx, \quad P_0(1, -1, 2), \quad \mathbf{v} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
16. $f(x, y, z) = x^2 + 2y^2 - 3z^2, \quad P_0(1, 1, 1), \quad \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
17. $g(x, y, z) = 3e^x \cos yz, \quad P_0(0, 0, 0), \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
18. $h(x, y, z) = \cos xy + e^{yz} + \ln zx, \quad P_0(1, 0, 1/2),$ $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

In Exercises 19–24, find the directions in which the functions increase most rapidly, and the directions in which they decrease most rapidly, at P_0. Then find the derivatives of the functions in these directions.

19. $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$
20. $f(x, y) = x^2 y + e^{xy} \sin y, \quad P_0(1, 0)$
21. $f(x, y, z) = (x/y) - yz, \quad P_0(4, 1, 1)$
22. $g(x, y, z) = xe^y + z^2, \quad P_0(1, \ln 2, 1/2)$
23. $f(x, y, z) = \ln xy + \ln yz + \ln xz, \quad P_0(1, 1, 1)$
24. $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z, \quad P_0(1, 1, 0)$

Tangent Lines to Level Curves

In Exercises 25–28, sketch the curve $f(x, y) = c$, together with ∇f and the tangent line at the given point. Then write an equation for the tangent line.

25. $x^2 + y^2 = 4, \quad (\sqrt{2}, \sqrt{2})$
26. $x^2 - y = 1, \quad (\sqrt{2}, 1)$
27. $xy = -4, \quad (2, -2)$
28. $x^2 - xy + y^2 = 7, \quad (-1, 2)$

Theory and Examples

29. Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f(1, -1)$ for which
    a. $D_{\mathbf{u}} f(1, -1)$ is largest
    b. $D_{\mathbf{u}} f(1, -1)$ is smallest
    c. $D_{\mathbf{u}} f(1, -1) = 0$
    d. $D_{\mathbf{u}} f(1, -1) = 4$
    e. $D_{\mathbf{u}} f(1, -1) = -3$
30. Let $f(x, y) = \dfrac{(x - y)}{(x + y)}$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ for which
    a. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is largest
    b. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is smallest
    c. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$
    d. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = -2$
    e. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$
31. **Zero directional derivative** In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?
32. **Zero directional derivative** In what directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?
33. Is there a direction $\mathbf{u}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14? Give reasons for your answer.

34. Changing temperature along a circle Is there a direction $\mathbf{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is $-3\,°\text{C/ft}$? Give reasons for your answer.

35. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in the direction of $-2\mathbf{j}$ is -3. What is the derivative of f in the direction of $-\mathbf{i} - 2\mathbf{j}$? Give reasons for your answer.

36. The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$.

a. What is ∇f at P? Give reasons for your answer.

b. What is the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$?

37. Directional derivatives and scalar components How is the derivative of a differentiable function $f(x, y, z)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$ related to the scalar component of $\nabla f|_{P_0}$ in the direction of $\mathbf{u}$? Give reasons for your answer.

38. Directional derivatives and partial derivatives Assuming that the necessary derivatives of $f(x, y, z)$ are defined, how are $D_{\mathbf{i}}f$, $D_{\mathbf{j}}f$, and $D_{\mathbf{k}}f$ related to f_x, f_y, and f_z? Give reasons for your answer.

39. Lines in the *xy*-plane Show that $A(x - x_0) + B(y - y_0) = 0$ is an equation for the line in the xy-plane through the point (x_0, y_0) normal to the vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$.

40. The algebra rules for gradients Given a constant k and the gradients

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

$$\nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k},$$

establish the algebra rules for gradients.

In Exercises 41–44, find a parametric equation for the line that is perpendicular to the graph of the given equation at the given point.

41. $x^2 + y^2 = 25, \quad (-3, 4)$

42. $x^2 + xy + y^2 = 3, \quad (2, -1)$

43. $x^2 + y^2 + z^2 = 14, \quad (3, -2, 1)$

44. $z = x^3 - xy^2, \quad (-1, 1, 0)$

Gradients and Directional Derivatives for Functions of More Than Three Variables

In Exercises 45–48, find ∇f at the given point.

45. $f(x, y, z, w) = \dfrac{x\sqrt{y}}{w} - x^2z^3, (2, 4, -1, 3)$

46. $f(x, y, z, w) = x^3 \sin y + w^2 \cos z, (-2, \pi, 0, 3)$

47. $f(x, y, z, s, t) = e^y \ln s + x^2t \tan z, \left(3, 0, \frac{\pi}{4}, e, 5\right)$

48. $f(x, y, z, s, t) = \dfrac{(x^2 + y^2)\arctan t}{z^2 s}, (-2, 1, -1, 2, 1)$

In Exercises 49–52, find the derivative of the function at P_0 in the direction of $\mathbf{v}$.

49. $f(x, y, z, w) = \dfrac{w \ln x}{y^2z^3}, \quad P_0(e^2, -2, 1, -3), \quad \mathbf{v} = \langle -1, 2, -2, 4\rangle$

50. $f(x, y, z, w) = (x - y)^2 + e^{z-w}, P_0(4, 2, 3, 1),$
$\mathbf{v} = \langle 1, 0, -2, 2\rangle$

51. $f(x, y, z, s, t) = s \arcsin(x + y) - t^2 \arctan(x - z),$
$P_0\left(0, \frac{1}{2}, -1, 1, -1\right), \mathbf{v} = \langle -1, 1, 0, 3, 5\rangle$

52. $f(x, y, z, s, t) = \sin tx + \cos sy - \dfrac{st}{z}, P_0\left(\frac{\pi}{4}, \frac{\pi}{6}, 2, 5, 1\right),$
$\mathbf{v} = \langle -3, 2, -2, 2, 2\rangle$

14.6 Tangent Planes and Differentials

In single-variable differential calculus, we saw how the derivative defined the tangent line to the graph of a differentiable function at a point on the graph. The tangent line then provided for a linearization of the function at the point. In this section, we will see analogously how the gradient defines the *tangent plane* to the level surface of a function $w = f(x, y, z)$ at a point on the surface. The tangent plane then provides for a linearization of f at the point and defines the total differential of the function.

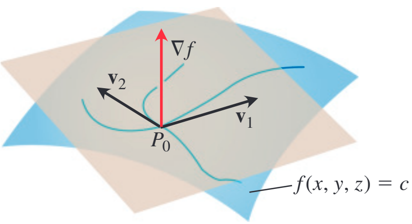

FIGURE 14.34 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0. The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0.

Tangent Planes and Normal Lines

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, we found in Equation (7) of the last section that

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Since f is constant along the curve $\mathbf{r}$, the derivative on the left-hand side of the equation is 0, so the gradient ∇f is orthogonal to the curve's velocity vector $\mathbf{r}'$.

Now let us restrict our attention to the curves that pass through a point P_0 (Figure 14.34). All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curves' tangent lines all lie in the plane through P_0 normal to ∇f. (assuming it is a nonzero vector). We now define this plane.