100. The heat equation An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

Show that $u(x, t) = \sin(\alpha x) \cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

101. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$. (*Hint:* Use Theorem 4 and show that f is not continuous at $(0, 0)$.)

102. Let $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

103. The Korteweg–de Vries equation

This nonlinear differential equation, which describes wave motion on shallow water surfaces, is given by

$$u_t + u_{xxx} + 12uu_x = 0.$$

Show that $u(x, t) = \text{sech}^2(x - t)$ satisfies the Korteweg–de Vries equation.

104. Show that $T = \dfrac{1}{\sqrt{x^2 + y^2}}$ satisfies the equation $T_{xx} + T_{yy} = T^3$.

14.4 The Chain Rule

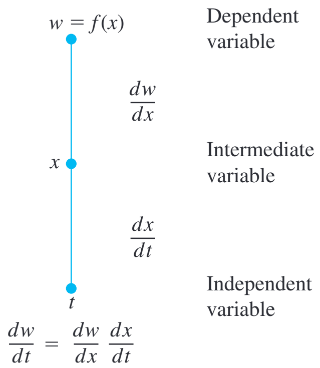
The Chain Rule for functions of a single variable studied in Section 3.6 says that if $w = f(x)$ is a differentiable function of x, and $x = g(t)$ is a differentiable function of t, then w is a differentiable function of t, and dw/dt can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

For this composite function $w(t) = f(g(t))$, we can think of t as the independent variable and $x = g(t)$ as the "intermediate variable" because t determines the value of x that in turn gives the value of w from the function f. We display the Chain Rule in a "dependency diagram" in the margin. Such diagrams capture which variables depend on which.

To find dw/dt, we read down the route from w to t, multiplying derivatives along the way.

Chain Rule



For functions of several variables the Chain Rule has more than one form, which depends on how many independent and intermediate variables are involved. However, once the variables are taken into account, the Chain Rule works in the same way we just discussed.

Functions of Two Variables

The Chain Rule formula for a differentiable function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

THEOREM 5—Chain Rule for Functions of One Independent Variable and Two Intermediate Variables

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composition $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Each of $\dfrac{\partial f}{\partial x}, \dfrac{\partial w}{\partial x}, f_x$ indicates the partial derivative of f with respect to x.

Proof The proof consists of showing that if x and y are differentiable at $t = t_0$, then w is differentiable at t_0 and

$$\frac{dw}{dt}(t_0) = \frac{\partial w}{\partial x}(P_0)\frac{dx}{dt}(t_0) + \frac{\partial w}{\partial y}(P_0)\frac{dy}{dt}(t_0),$$

where $P_0 = (x(t_0), y(t_0))$.

Let Δx, Δy, and Δw be the increments that result from changing t from t_0 to $t_0 + \Delta t$. Since f is differentiable (see the definition in Section 14.3),

$$\Delta w = \frac{\partial w}{\partial x}(P_0)\,\Delta x + \frac{\partial w}{\partial y}(P_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. To find dw/dt, we divide this equation through by Δt and let Δt approach zero (therefore, Δx and Δy approach zero as well since the fact that $x(t)$ and $y(t)$ are differentiable implies that they are continuous). The division gives

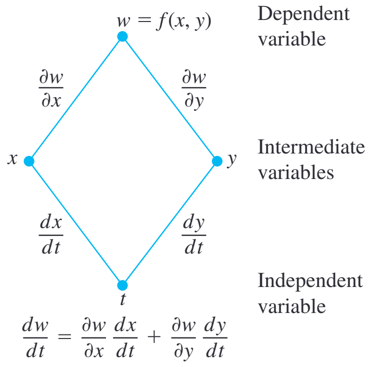
$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}(P_0)\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}(P_0)\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}.$$

Letting Δt approach zero gives

$$\begin{aligned}\frac{dw}{dt}(t_0) &= \lim_{\Delta t \to 0}\frac{\Delta w}{\Delta t} \\ &= \frac{\partial w}{\partial x}(P_0)\frac{dx}{dt}(t_0) + \frac{\partial w}{\partial y}(P_0)\frac{dy}{dt}(t_0) + 0 \cdot \frac{dx}{dt}(t_0) + 0 \cdot \frac{dy}{dt}(t_0).\end{aligned}$$ ■

To remember the Chain Rule, picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



Often we write $\partial w/\partial x$ for the partial derivative $\partial f/\partial x$, so we can rewrite the Chain Rule in Theorem 5 in the form

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

However, the meaning of the dependent variable w is different on each side of the preceding equation. On the left-hand side, it refers to the composite function $w = f(x(t), y(t))$ as a function of the single variable t. On the right-hand side, it refers to the function $w = f(x, y)$ as a function of the two variables x and y. Moreover, the single derivatives dw/dt, dx/dt, and dy/dt are being evaluated at a point t_0, whereas the partial derivatives $\partial w/\partial x$ and $\partial w/\partial y$ are being evaluated at the point (x_0, y_0), with $x_0 = x(t_0)$ and $y_0 = y(t_0)$. With that understanding, we will use both of these forms interchangeably throughout the text whenever no confusion will arise.

The **dependency diagram** on the preceding page provides a convenient way to remember the Chain Rule. The "true" independent variable in the composite function is t, whereas x and y are *intermediate variables* (controlled by t) and w is the dependent variable.

A more precise notation for the Chain Rule shows where the various derivatives in Theorem 5 are evaluated:

$$\frac{dw}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0),$$

or, using another notation,

$$\left.\frac{dw}{dt}\right|_{t_0} = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}\left.\frac{dx}{dt}\right|_{t_0} + \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}\left.\frac{dy}{dt}\right|_{t_0}.$$

EXAMPLE 1 Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\cdot\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\cdot\frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}(2\cos 2t) = \cos 2t.$$

In either case, at the given value of t,

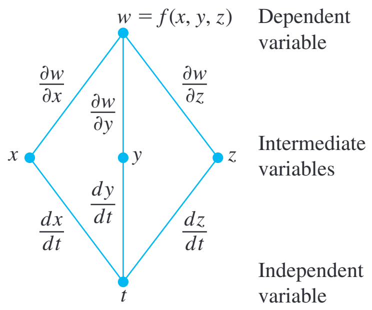
$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2\frac{\pi}{2}\right) = \cos \pi = -1.$$ ■

Functions of Three Variables

You can probably predict the Chain Rule for functions of three intermediate variables, as it involves adding the expected third term to the two-variable formula.

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

THEOREM 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables
If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

The proof is identical to the proof of Theorem 5, except that there are now three intermediate variables instead of two. The dependency diagram we use for remembering the new equation is similar as well, with three routes from w to t.

EXAMPLE 2 Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \qquad \text{Substitute for intermediate variables.} \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,
\end{aligned}$$

so

$$\left.\frac{dw}{dt}\right|_{t=0} = 1 + \cos(0) = 2.$$

For a physical interpretation of change along a curve, think of an object whose position is changing with time t. If $w = T(x, y, z)$ is the temperature at each point (x, y, z) along a curve C with parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$, then the composite function $w = T(x(t), y(t), z(t))$ represents the temperature relative to t along the curve. The derivative dw/dt is then the instantaneous rate of change of temperature due to the motion along the curve, as calculated in Theorem 6.

Functions Defined on Surfaces

If we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on Earth's surface, we might prefer to think of x, y, and z as functions of the variables r and s that give the points' longitudes and latitudes. If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of r and s with the composite function

$$w = f(g(r, s), h(r, s), k(r, s)).$$

Under the conditions stated below, w has partial derivatives with respect to both r and s that can be calculated in the following way.

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The first of these equations can be derived from the Chain Rule in Theorem 6 by holding s fixed and treating r as t. The second can be derived in the same way, holding r fixed and treating s as t. The dependency diagrams for both equations are shown in Figure 14.22.

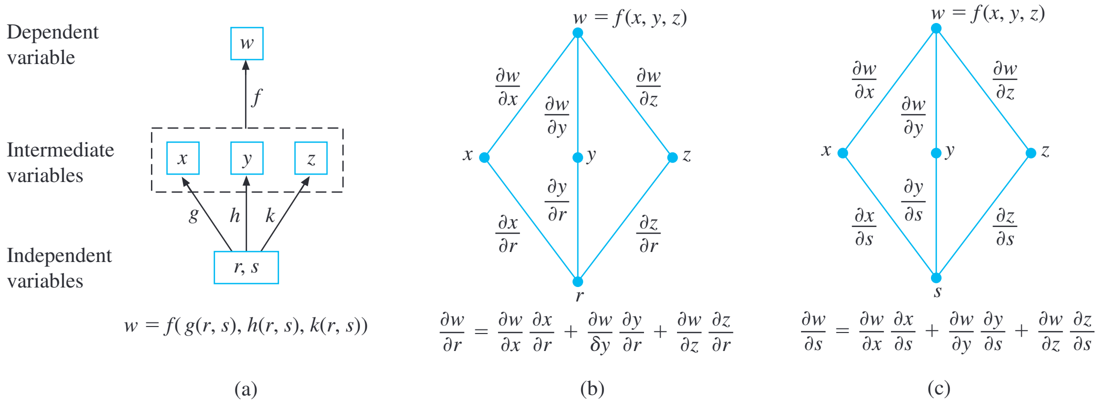


FIGURE 14.22 Composite function and dependency diagrams for Theorem 7.

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution Using the formulas in Theorem 7, we find

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \qquad \text{Substitute for intermediate variable } z.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.
\end{aligned}$$

If f is a function of two intermediate variables instead of three, each equation in Theorem 7 becomes correspondingly one term shorter.

> If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then
> $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Chain Rule

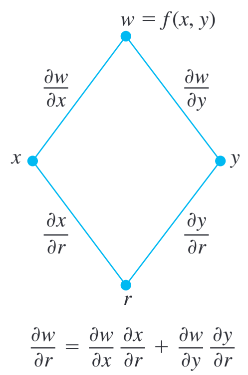


FIGURE 14.23 Dependency diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

Figure 14.23 shows the dependency diagram for the first of these equations. The diagram for the second equation is similar; just replace r with s.

EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

Solution The preceding discussion gives the following.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 4r\end{aligned} \qquad \begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= 4s\end{aligned}$$

Substitute for the intermediate variables.

If f is a function of a single intermediate variable x, our equations are even simpler.

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

Chain Rule

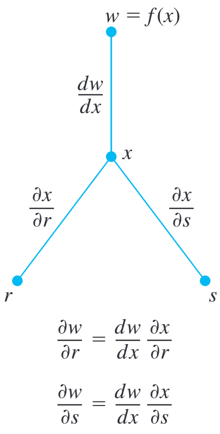


FIGURE 14.24 Dependency diagram for differentiating f as a composite function of r and s with one intermediate variable.

In this case, we use the ordinary (single-variable) derivative, dw/dx. The dependency diagram is shown in Figure 14.24.

Implicit Differentiation Revisited

The two-variable Chain Rule in Theorem 5 leads to a formula that takes some of the algebra out of implicit differentiation. Suppose that

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, h(x)) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Since $w = F(x, h(x)) = 0$, the derivative dw/dx must be zero. Computing the derivative from the Chain Rule (dependency diagram in Figure 14.25), we find

$$\begin{aligned}0 = \frac{dw}{dx} &= F_x\frac{dx}{dx} + F_y\frac{dy}{dx} \\ &= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.\end{aligned}$$

Theorem 5 with $t = x$ and $f = F$

If $F_y = \partial w/\partial y \neq 0$, we can solve this equation for dy/dx to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

We state this result formally.

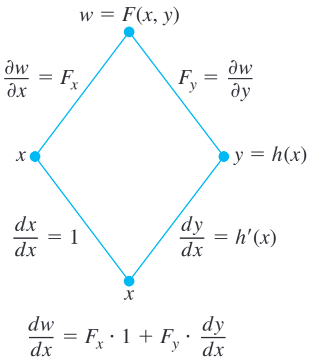


FIGURE 14.25 Dependency diagram for differentiating $w = F(x, y)$ with respect to x. Setting $dw/dx = 0$ leads to a simple computational formula for implicit differentiation (Theorem 8).

THEOREM 8—A Formula for Implicit Differentiation
Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then, at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

EXAMPLE 5 Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Solution Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

This calculation is significantly shorter than a single-variable calculation using implicit differentiation. ■

The result in Theorem 8 is easily extended to three variables. Suppose that the equation $F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$. Then, for all (x, y) in the domain of f, we have $F(x, y, f(x, y)) = 0$. Assuming that F and f are differentiable functions, we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to the independent variable x:

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x}, \end{aligned}$$

y is constant when we differentiate with respect to x.

so

$$F_x + F_z\frac{\partial z}{\partial x} = 0.$$

A similar calculation for differentiating with respect to the independent variable y gives

$$F_y + F_z\frac{\partial z}{\partial y} = 0.$$

Whenever $F_z \neq 0$, we can solve these last two equations for the partial derivatives of $z = f(x, y)$ to obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \qquad (2)$$

An important result from advanced calculus, called the **Implicit Function Theorem**, states the conditions for which our results in Equations (2) are valid. If the partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0), and if for some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by Equations (2).

EXAMPLE 6 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Solution Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad \text{and} \qquad F_z = 2z + xye^{xz} + \cos y.$$

Since $F(0,0,0)=0$, $F_z(0,0,0)=1\neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x,y,z)=0$ defines z as a differentiable function of x and y near the point $(0,0,0)$. From Equations (2),

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+zye^{xz}}{2z+xye^{xz}+\cos y}\quad\text{and}\quad\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{e^{xz}-z\sin y}{2z+xye^{xz}+\cos y}.$$

At $(0,0,0)$ we find

$$\frac{\partial z}{\partial x}=-\frac{0}{1}=0\quad\text{and}\quad\frac{\partial z}{\partial y}=-\frac{1}{1}=-1.$$

Functions of Many Variables

We have seen several different forms of the Chain Rule in this section, but each one is just a special case of one general formula. When solving particular problems, it may help to draw the appropriate dependency diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of the dependent variable with respect to the selected independent variable, start at the dependent variable and read down each route of the dependency diagram to the independent variable, calculating and multiplying the derivatives along each route. Then add the products found for the different routes.

In general, suppose that $w=f(x_1,x_2,\dots,x_n)$ is a differentiable function of the intermediate variables $x_1,x_2,\dots,x_n$ (a finite set) and that $x_1,x_2,\dots,x_n$ are differentiable functions of the independent variables $t_1,t_2,\dots,t_m$ (another finite set). Then w is a differentiable function of the variables $t_1,t_2,\dots,t_m$, and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial t_i}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\cdots+\frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_i}\quad\text{for } i=1,2,\dots,m.$$

One way to remember this equation is to think of the right-hand side as the dot product of two n-dimensional vectors:

$$\frac{\partial w}{\partial t_i}=\underbrace{\left\langle\frac{\partial w}{\partial x_1},\frac{\partial w}{\partial x_2},\dots,\frac{\partial w}{\partial x_n}\right\rangle}_{\substack{\text{Derivatives of } w \text{ with}\\ \text{respect to the}\\ \text{intermediate variables}}}\cdot\underbrace{\left\langle\frac{\partial x_1}{\partial t_i},\frac{\partial x_2}{\partial t_i},\dots,\frac{\partial x_n}{\partial t_i}\right\rangle}_{\substack{\text{Derivatives of the}\\ \text{intermediate variables with}\\ \text{respect to the selected}\\ \text{independent variable}}}.$$

The first vector describes how w changes in various directions, while the second vector indicates the velocity vector of $\mathbf{x}(t_i)=\langle x_1(t_i),x_2(t_i),\dots,x_n(t_i)\rangle$. These concepts will be studied further in the next section.

EXERCISES 14.4

Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then **(b)** evaluate dw/dt at the given value of t.

1. $w=x^2+y^2,\quad x=\cos t,\quad y=\sin t;\quad t=\pi$

2. $w=x^2+y^2,\quad x=\cos t+\sin t,\quad y=\cos t-\sin t;\quad t=0$

3. $w=\dfrac{x}{z}+\dfrac{y}{z},\quad x=\cos^2 t,\quad y=\sin^2 t,\quad z=1/t;\quad t=3$

4. $w=\ln(x^2+y^2+z^2),\quad x=\cos t,\quad y=\sin t,\quad z=4\sqrt{t};\quad t=3$

5. $w=2ye^x-\ln z,\quad x=\ln(t^2+1),\quad y=\tan^{-1}t,\quad z=e^t;\quad t=1$

6. $w=z-\sin xy,\quad x=t,\quad y=\ln t,\quad z=e^{t-1};\quad t=1$

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u,v).

7. $z=4e^x\ln y,\quad x=\ln(u\cos v),\quad y=u\sin v;\quad (u,v)=(2,\pi/4)$

8. $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u\sin v;$
$(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

9. $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

10. $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$
$z = ue^v; \quad (u, v) = (-2, 0)$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z).

11. $u = \dfrac{p - q}{q - r}, \quad p = x + y + z, \quad q = x - y + z,$
$r = x + y - z; \quad (x, y, z) = (\sqrt{3}, 2, 1)$

12. $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = 1/z;$
$(x, y, z) = (\pi/4, 1/2, -1/2)$

Using a Dependency Diagram

In Exercises 13–24, draw a dependency diagram and write a Chain Rule formula for each derivative.

13. $\dfrac{dz}{dt}$ for $z = f(x, y), \quad x = g(t), \quad y = h(t)$

14. $\dfrac{dz}{dt}$ for $z = f(u, v, w), \quad u = g(t), \quad v = h(t), \quad w = k(t)$

15. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = h(x, y, z), \quad x = f(u, v), \quad y = g(u, v),$
$z = k(u, v)$

16. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = f(r, s, t), \quad r = g(x, y), \quad s = h(x, y),$
$t = k(x, y)$

17. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = g(x, y), \quad x = h(u, v), \quad y = k(u, v)$

18. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = g(u, v), \quad u = h(x, y), \quad v = k(x, y)$

19. $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$ for $z = f(x, y), \quad x = g(t, s), \quad y = h(t, s)$

20. $\dfrac{\partial y}{\partial r}$ for $y = f(u), \quad u = g(r, s)$

21. $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$ for $w = g(u), \quad u = h(s, t)$

22. $\dfrac{\partial w}{\partial p}$ for $w = f(x, y, z, v), \quad x = g(p, q), \quad y = h(p, q),$
$z = j(p, q), \quad v = k(p, q)$

23. $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ for $w = f(x, y), \quad x = g(r), \quad y = h(s)$

24. $\dfrac{\partial w}{\partial s}$ for $w = g(x, y), \quad x = h(r, s, t), \quad y = k(r, s, t)$

Implicit Differentiation

Assuming that the equations in Exercises 25–30 define y as a differentiable function of x, use Theorem 8 to find the value of dy/dx at the given point.

25. $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

26. $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$

27. $x^2 + xy + y^2 - 7 = 0, \quad (1, 2)$

28. $xe^y + \sin xy + y - \ln 2 = 0, \quad (0, \ln 2)$

29. $(x^3 - y^4)^6 + \ln(x^2 + y) = 1, \quad (-1, 0)$

30. $xe^{x^2 y} - ye^x = x + y - 2, \quad (1, 1)$

Find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 31–34.

31. $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

32. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0, \quad (2, 3, 6)$

33. $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0, \quad (\pi, \pi, \pi)$

34. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, \quad (1, \ln 2, \ln 3)$

Finding Partial Derivatives at Specified Points

35. Find $\partial w/\partial r$ when $r = 1, s = -1$ if $w = (x + y + z)^2$, $x = r - s, y = \cos(r + s), z = \sin(r + s)$.

36. Find $\partial w/\partial v$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u, y = u + v, z = \cos u$.

37. Find $\partial w/\partial v$ when $u = 0, v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1, y = 2u + v - 2$.

38. Find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x\sin y$, $x = u^2 + v^2, y = uv$.

39. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = \ln 2, v = 1$ if $z = 5\tan^{-1} x$ and $x = e^u + \ln v$.

40. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = 1, v = -2$ if $z = \ln q$ and $q = \sqrt{v + 3}\tan^{-1} u$.

Theory and Examples

41. Assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

42. Assume that $w = f\left(ts^2, \dfrac{s}{t}\right)$, $\dfrac{\partial f}{\partial x}(x, y) = xy$, and $\dfrac{\partial f}{\partial y}(x, y) = \dfrac{x^2}{2}$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

43. Assume that $z = f(x, y), x = g(t), y = h(t), f_x(2, -1) = 3$, and $f_y(2, -1) = -2$. If $g(0) = 2, h(0) = -1, g'(0) = 5$, and $h'(0) = -4$, find $\left.\dfrac{dz}{dt}\right|_{t=0}$.

44. Assume that $z = f(x, y)^2, x = g(t), y = h(t), f_x(1, 0) = -1$, $f_y(1, 0) = 1$, and $f(1, 0) = 2$. If $g(3) = 1, h(3) = 0$, $g'(3) = -3$, and $h'(3) = 4$, find $\left.\dfrac{dz}{dt}\right|_{t=3}$.

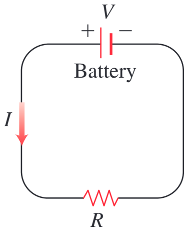
45. Assume that $z = f(w), w = g(x, y), x = 2r^3 - s^2$, and $y = re^s$. If $g_x(2, 1) = -3, g_y(2, 1) = 2, f'(7) = -1$, and $g(2, 1) = 7$, find $\left.\dfrac{\partial z}{\partial r}\right|_{r=1, s=0}$ and $\left.\dfrac{\partial z}{\partial s}\right|_{r=1, s=0}$.

46. Assume that $z = \ln(f(w)), w = g(x, y), x = \sqrt{r - s}$, and $y = r^2 s$. If $g_x(2, -9) = -1, g_y(2, -9) = 3, f'(-2) = 2$, $f(-2) = 5$, and $g(2, -9) = -2$, find $\left.\dfrac{\partial z}{\partial r}\right|_{r=3, s=-1}$ and $\left.\dfrac{\partial z}{\partial s}\right|_{r=3, s=-1}$.

47. Changing voltage in a circuit The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

to find how the current is changing at the instant when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.



48. Changing dimensions in a box The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, $a = 1$ m, $b = 2$ m, $c = 3$ m, $da/dt = db/dt = 1$ m/sec, and $dc/dt = -3$ m/sec. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

49. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

50. Polar coordinates Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta.$$

b. Solve the equations in part (a) to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial\theta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

51. Laplace equations Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

52. Laplace equations Let $w = f(u) + g(v)$, where $u = x + iy$, $v = x - iy$, and $i = \sqrt{-1}$. Show that w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$ if all the necessary functions are differentiable.

53. Extreme values on a helix Suppose that the partial derivatives of a function $f(x, y, z)$ at points on the helix $x = \cos t$, $y = \sin t$, $z = t$ are

$$f_x = \cos t, \quad f_y = \sin t, \quad f_z = t^2 + t - 2.$$

At what points on the curve, if any, can f take on extreme values?

54. A space curve Let $w = x^2 e^{2y}\cos 3z$. Find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x = \cos t$, $y = \ln(t + 2)$, $z = t$.

55. Temperature on a circle Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

a. Find where the maximum and minimum temperatures on the circle occur by examining the derivatives dT/dt and d^2T/dt^2.

b. Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

56. Temperature on an ellipse Let $T = g(x, y)$ be the temperature at the point (x, y) on the ellipse

$$x = 2\sqrt{2}\cos t, \qquad y = \sqrt{2}\sin t, \qquad 0 \le t \le 2\pi,$$

and suppose that

$$\frac{\partial T}{\partial x} = y, \qquad \frac{\partial T}{\partial y} = x.$$

a. Locate the maximum and minimum temperatures on the ellipse by examining dT/dt and d^2T/dt^2.

b. Suppose that $T = xy - 2$. Find the maximum and minimum values of T on the ellipse.

57. The temperature $T = T(x, y)$ in °C at point (x, y) satisfies $T_x(1, 2) = 3$ and $T_y(1, 2) = -1$. If $x = e^{2t-2}$ cm and $y = 2 + \ln t$ cm, find the rate at which the temperature T changes when $t = 1$ sec.

58. A bug crawls on the surface $z = x^2 - y^2$ directly above a path in the xy-plane given by $x = f(t)$ and $y = g(t)$. If $f(2) = 4$, $f'(2) = -1$, $g(2) = -2$, and $g'(2) = -3$, then at what rate is the bug's elevation z changing when $t = 2$?

Differentiating Integrals Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\,dt,$$

then $F'(x) = \int_a^b g_x(t, x)\,dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x)\,dt$$

by letting

$$G(u, x) = \int_a^u g(t, x)\,dt,$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 59 and 60.

59. $F(x) = \displaystyle\int_0^{x^2} \sqrt{t^4 + x^3}\,dt$

60. $F(x) = \displaystyle\int_{x^2}^1 \sqrt{t^3 + x^2}\,dt$

61. Water is flowing into a tank in the form of a right-circular cylinder at the rate of $(4/5)\pi$ ft³/min. The tank is stretching in such a way that even though it remains cylindrical, its radius is increasing at the rate of 0.002 ft/min. How fast is the surface of the water rising when the radius is 2 ft and the volume of water in the tank is 20π ft³?

62. Suppose f is a differentiable function of x, y, and z and $u = f(x, y, z)$. Then if $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\phi$, express $\partial u/\partial r$, $\partial u/\partial\phi$, and $\partial u/\partial\theta$ in terms of $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$.

63. At a given instant, the length of one leg of a right triangle is 10 ft, and it is increasing at the rate of 1 ft/min, and the length of the other leg of the right triangle is 12 ft, and it is decreasing at the rate of 2 ft/min. Find the rate of change of the measure of the acute angle opposite the leg of length 12 ft at the given instant.