takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x,y)\to(0,0)} x^2/(x^2+y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \to 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. Example 5 illustrates this point. In polar coordinates, $f(x,y) = (2x^2y)/(x^4+y^2)$ becomes

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + \sin^2\theta}$$

for $r \neq 0$. If we hold θ constant and let $r \to 0$, the limit is 0. On the path $y = x^2$, however, we have $r\sin\theta = r^2\cos^2\theta$ and

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + (r\cos^2\theta)^2} \\ &= \frac{2r\cos^2\theta\sin\theta}{2r^2\cos^4\theta} = \frac{r\sin\theta}{r^2\cos^2\theta} = 1. \end{aligned}$$

In Exercises 65–70, find the limit of f as $(x,y) \to (0,0)$ or show that the limit does not exist.

65. $f(x,y) = \dfrac{x^3 - xy^2}{x^2+y^2}$

66. $f(x,y) = \cos\left(\dfrac{x^3-y^3}{x^2+y^2}\right)$

67. $f(x,y) = \dfrac{y^2}{x^2+y^2}$

68. $f(x,y) = \dfrac{2x}{x^2+x+y^2}$

69. $f(x,y) = \tan^{-1}\left(\dfrac{|x|+|y|}{x^2+y^2}\right)$

70. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

In Exercises 71 and 72, define $f(0,0)$ in a way that extends f to be continuous at the origin.

71. $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$

72. $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

Using the Limit Definition

Each of Exercises 73–78 gives a function $f(x,y)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x,y),

$$\sqrt{x^2+y^2} < \delta \quad \Rightarrow \quad |f(x,y) - f(0,0)| < \varepsilon.$$

73. $f(x,y) = x^2+y^2, \quad \varepsilon = 0.01$

74. $f(x,y) = y/(x^2+1), \quad \varepsilon = 0.05$

75. $f(x,y) = (x+y)/(x^2+1), \quad \varepsilon = 0.01$

76. $f(x,y) = (x+y)/(2+\cos x), \quad \varepsilon = 0.02$

77. $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ and $f(0,0) = 0, \quad \varepsilon = 0.04$

78. $f(x,y) = \dfrac{x^3+y^4}{x^2+y^2}$ and $f(0,0) = 0, \quad \varepsilon = 0.02$

Each of Exercises 79–82 gives a function $f(x,y,z)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x,y,z),

$$\sqrt{x^2+y^2+z^2} < \delta \quad \Rightarrow \quad |f(x,y,z) - f(0,0,0)| < \varepsilon.$$

79. $f(x,y,z) = x^2+y^2+z^2, \quad \varepsilon = 0.015$

80. $f(x,y,z) = xyz, \quad \varepsilon = 0.008$

81. $f(x,y,z) = \dfrac{x+y+z}{x^2+y^2+z^2+1}, \quad \varepsilon = 0.015$

82. $f(x,y,z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \varepsilon = 0.03$

83. Show that $f(x,y,z) = x+y-z$ is continuous at every point (x_0, y_0, z_0).

84. Show that $f(x,y,z) = x^2+y^2+z^2$ is continuous at the origin.

14.3 Partial Derivatives

The calculus of several variables is similar to single-variable calculus applied to several variables, one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the familiar rules for differentiating functions of a single variable. The idea of *differentiability* for functions of several variables requires more than the existence of the partial derivatives, because a point can be approached from many different directions. However, we will see that differentiable functions of several variables behave similarly to differentiable single-variable functions. In particular, they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x,y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x,y)$ in the curve $z = f(x, y_0)$ (Figure 14.16). This curve is the graph

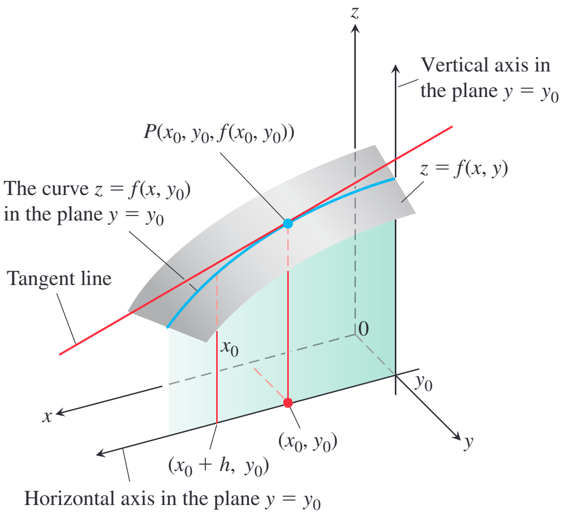


FIGURE 14.16 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x; the vertical coordinate is z. The y-value is held constant at y_0, so y is not a variable.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives, we use the symbol ∂ rather than the d previously used. In the definition, h represents a real number, positive or negative.

DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is the same as the ordinary derivative of $f(x, y_0)$ at the point x_0:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x = x_0}.$$

A variety of notations are used to denote the partial derivative at a point (x_0, y_0), including

$$\frac{\partial f}{\partial x}(x_0, y_0), \qquad f_x(x_0, y_0), \qquad \text{and} \qquad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}.$$

When we do not specify a specific point (x_0, y_0) at which the partial derivative is being evaluated, then the partial derivative becomes a *function* whose domain is the points where the partial derivative exists. Notations for this function include

$$\frac{\partial f}{\partial x}, \qquad f_x, \qquad \text{and} \qquad \frac{\partial z}{\partial x}.$$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). (In Figure 14.16 this slope is negative.) The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f/\partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0.

DEFINITION The **partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

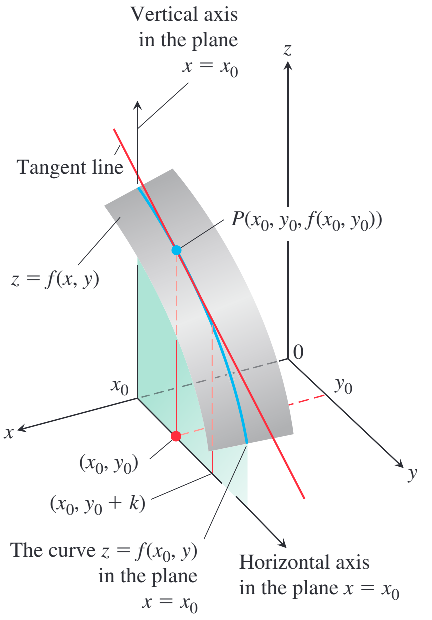


FIGURE 14.17 The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Figure 14.17) is the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0.

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x:

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (Figure 14.18). Is the plane they determine tangent to the surface at P? We will see that it is for the *differentiable* functions defined at the end of this section, and we will learn how to find the tangent plane in Section 14.6. First we have to better understand partial derivatives.

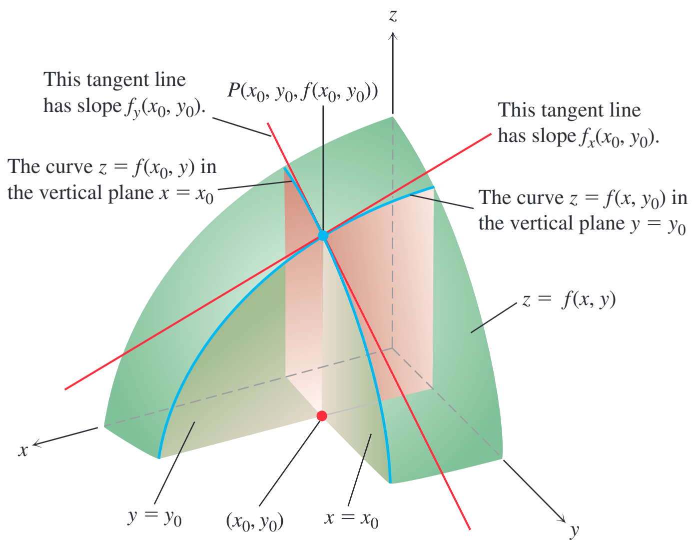


FIGURE 14.18 Figures 14.16 and 14.17 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Calculations

The definitions of $\partial f/\partial x$ and $\partial f/\partial y$ give us two different ways of differentiating f at a point: with respect to x in the usual way while treating y as a constant, and with respect to y in the usual way while treating x as a constant. As the following examples show, the values of these partial derivatives are usually different at a given point (x_0, y_0).

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y) \\ &= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.\end{aligned}$$ ■

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we use the quotient rule to get

$$\begin{aligned}f_x &= \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.\end{aligned}$$

With x held constant and again applying the quotient rule, we get

$$\begin{aligned}f_y &= \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.\end{aligned}$$ ■

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

EXAMPLE 4 Find $\partial z/\partial x$ assuming that the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \qquad \text{With } y \text{ constant, } \frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}.$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

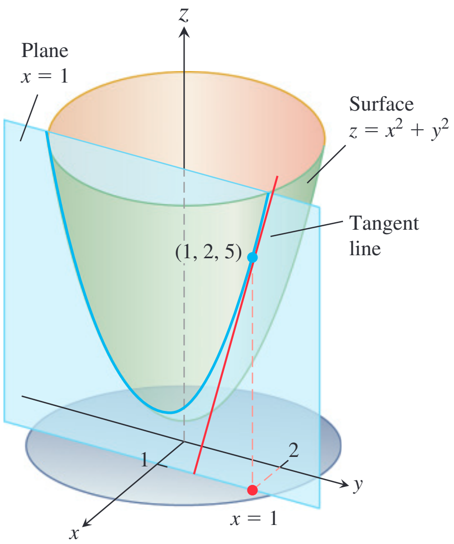


FIGURE 14.19 The tangent line to the curve of intersection of the plane $x = 1$ and the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$ (Example 5).

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent line to the parabola at $(1, 2, 5)$ (Figure 14.19).

Solution The parabola lies in a plane parallel to the yz-plane, and the slope is the value of the partial derivative $\partial z/\partial y$ at $(1, 2)$:

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are similar to the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

EXAMPLE 6 If x, y, and z are independent variables and

$$f(x, y, z) = x\sin(y + 3z),$$

then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x\sin(y + 3z)] = x\frac{\partial}{\partial z}\sin(y + 3z) \qquad x \text{ held constant}$$

$$= x\cos(y + 3z)\frac{\partial}{\partial z}(y + 3z) \qquad \text{Chain rule}$$

$$= 3x\cos(y + 3z). \qquad y \text{ held constant}$$

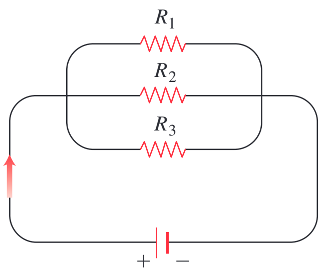


FIGURE 14.20 Resistors arranged this way are said to be connected in parallel (Example 7). Each resistor lets a portion of the current through. Their equivalent resistance R is calculated with the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

EXAMPLE 7 If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(Figure 14.20). Find the value of $\partial R/\partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution To find $\partial R/\partial R_2$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2:

$$\begin{aligned}
\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) &= \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\\
-\frac{1}{R^2}\frac{\partial R}{\partial R_2} &= 0 - \frac{1}{R_2{}^2} + 0\\
\frac{\partial R}{\partial R_2} &= \frac{R^2}{R_2{}^2} = \left(\frac{R}{R_2}\right)^2.
\end{aligned}$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3+2+1}{90} = \frac{6}{90} = \frac{1}{15},$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Thus at the given values, a small change in the resistance R_2 leads to a change in R about one-ninth as large. ■

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. If the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0), however, then f is continuous at (x_0, y_0), as we see at the end of this section.

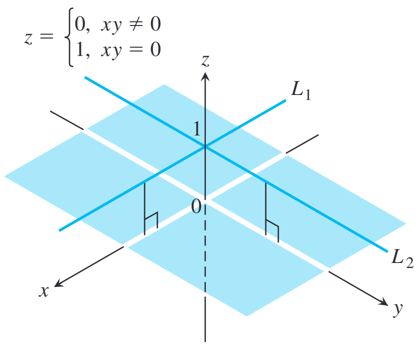


FIGURE 14.21 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 (lying 1 unit above the xy-plane) and the four open quadrants of the xy-plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

(a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

(b) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = 0$.

(c) Prove that f is not continuous at the origin.

(d) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Solution

(a) Since $f(x, y)$ is zero at every point on the line $y = x$ (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(b) Since $f(x, y)$ takes the constant value 1 at every point on the line $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=0} = \lim_{(x,y)\to(0,0)} 1 = 1.$$

(c) By the two-path test, f has no limit as (x, y) approaches $(0, 0)$. Consequently, f is not continuous at $(0, 0)$.

(d) To find $\partial f/\partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x, and the graph of f is the line L_1 in Figure 14.21. The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0, 0)$. Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y, so $\partial f/\partial y = 0$ at $(0, 0)$. ■

What Example 8 suggests is that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives. We define differentiability for functions of two variables (which is somewhat more complicated than for single-variable functions) at the end of this section and then revisit the connection to continuity.

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x\,\partial y} \text{ or } f_{yx}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y\,\partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$$\frac{\partial^2 f}{\partial x\,\partial y}$$ Differentiate first with respect to y, then with respect to x.

$$f_{yx} = (f_y)_x$$ Means the same thing

HISTORICAL BIOGRAPHY
Pierre-Simon Laplace
(1749–1827)
www.bit.ly/2NdfIV5

EXAMPLE 9 If $f(x, y) = x\cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y\,\partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x\,\partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$ ■

The Mixed Derivative Theorem

You may have noticed that the "mixed" second-order partial derivatives

$$\frac{\partial^2 f}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x\,\partial y}$$

in Example 9 are equal. This is not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous, as stated in the following theorem. However, the mixed derivatives can be different when the continuity conditions are not satisfied (see Exercise 82).

THEOREM 2—The Mixed Derivative Theorem
If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

HISTORICAL BIOGRAPHY
Alexis Clairaut
(1713–1765)
www.bit.ly/2Rfl7yc

Theorem 2 is also known as Clairaut's Theorem, after the French mathematician Alexis Clairaut, who discovered it. A proof is given in Appendix A.8. Theorem 2 says that to calculate a mixed second-order derivative, we may differentiate in either order, provided the continuity conditions are satisfied. This ability to proceed in different order sometimes simplifies our calculations.

EXAMPLE 10 Find $\dfrac{\partial^2 w}{\partial x\,\partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x\,\partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y\,\partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x\,\partial y = 1$ as well, but with more work. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0). ■

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x\,\partial y^2} = f_{yyx},$$
$$\frac{\partial^4 f}{\partial x^2\,\partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$ ■

Differentiability

The concept of differentiability for functions of several variables is more complicated than for single-variable functions, because a point in the domain can be approached from many directions and along any path, not just from the left or from the right. The existence of both partial derivatives at a point (x_0, y_0) is not by itself even enough to show continuity at (x_0, y_0), as we saw in Example 8. The differentiability of f is instead based on the idea that a linear function gives a good model of a differentiable function near a point.

In Section 3.11, we saw that a differentiable function f can be approximated near a point x_0 by its linearization,

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

This formula allows us to find a linear function L, a function whose graph is a straight line, such that L closely approximates f near x_0. This can be done whenever f is differentiable, even when f itself is described by a very complicated formula. Approximations are much more useful and meaningful when they are accompanied by information on their accuracy. In Section 3.11, Equation (1), we saw that a differentiable function f satisfies

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \varepsilon(x - x_0),$$

where $\varepsilon \to 0$ as $x \to x_0$. Framed in terms of approximating f by L, this becomes

$$f(x) - L(x) = \varepsilon(x - x_0), \tag{1}$$

where again $\varepsilon \to 0$ as $x \to x_0$.

Rather than being a consequence of the definition, the differentiability for a function of two variables $f(x, y)$ is defined to mean that f can be approximated by a linear function. The approximating linear function $L(x, y)$ for $f(x, y)$ near the point (x_0, y_0) takes the form

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

and the graph of L is a plane, called the *tangent plane*, that approximates the graph of f near (x_0, y_0). Notice that $L(x_0, y_0) = f(x_0, y_0)$, so the functions L and f coincide at (x_0, y_0). Moreover the partial derivatives of L and f are also equal at (x_0, y_0). We will study tangent planes in detail in Section 14.6.

We now specify how closely f is approximated by L at (x_0, y_0). Extending the formula for single variable functions in Equation (1), we require that the difference between f and L satisfies

$$f(x, y) - L(x, y) = \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0), \tag{2}$$

where both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.

If we insert the formula for $L(x, y)$ into Equation (2) we see that

$$\begin{aligned} f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \varepsilon_1(x - x_0) \\ + \varepsilon_2(y - y_0). \end{aligned}$$

Setting $\Delta x = x - x_0, \Delta y = y - y_0$, and $\Delta z = f(x, y) - f(x_0, y_0)$, we get

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y.$$

Based on these ideas, we now state the formal definition of differentiability, which captures the idea that f is well approximated by L.

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and if $\Delta z = f(x, y) - f(x_0, y_0)$ satisfies

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where $\Delta x = x - x_0, \Delta y = y - y_0$, and both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$. We call the function f **differentiable** if it is differentiable at every point in its domain, and we then say that its graph is a **smooth surface**.

Differentiability is defined in a similar way for functions of more than two variables.

The following theorem (proved in Appendix A.8) and its accompanying corollary tell us that functions with *continuous* first partial derivatives at (x_0, y_0) are differentiable there, and they are closely approximated locally by a linear function. We study this approximation in Section 14.6.

THEOREM 3—The Increment Theorem for Functions of Two Variables
Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

In many cases the partial derivatives are defined and continuous at every point in the domain of f. We then have the following Corollary.

Corollary of Theorem 3
If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

If $z = f(x, y)$ is differentiable, then the definition of differentiability ensures that $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ approaches 0 as Δx and Δy approach 0. This tells us that a function of two variables is continuous at every point where it is differentiable.

THEOREM 4—Differentiable Implies Continuous
If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

As we can see from Corollary 3 and Theorem 4, a function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0). Remember, however, that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist, as we saw in Example 8. Existence alone of the partial derivatives at that point is not enough, but continuity of the partial derivatives guarantees differentiability.

EXERCISES 14.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$

2. $f(x, y) = x^2 - xy + y^2$

3. $f(x, y) = (x^2 - 1)(y + 2)$

4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

5. $f(x, y) = (xy - 1)^2$

6. $f(x, y) = (2x - 3y)^3$

7. $f(x, y) = \sqrt{x^2 + y^2}$

8. $f(x, y) = (x^3 + (y/2))^{2/3}$

9. $f(x, y) = 1/(x + y)$

10. $f(x, y) = x/(x^2 + y^2)$

11. $f(x, y) = (x + y)/(xy - 1)$

12. $f(x, y) = \tan^{-1}(y/x)$

13. $f(x, y) = e^{(x+y+1)}$

14. $f(x, y) = e^{-x}\sin(x + y)$

15. $f(x, y) = \ln(x + y)$

16. $f(x, y) = e^{xy}\ln y$

17. $f(x, y) = \sin^2(x - 3y)$

18. $f(x, y) = \cos^2(3x - y^2)$

19. $f(x, y) = x^y$

20. $f(x, y) = \log_y x$

21. $f(x, y) = \int_x^y g(t)\,dt$ (g continuous for all t)

22. $f(x, y) = \sum_{n=0}^{\infty} (xy)^n$ $(|xy| < 1)$

In Exercises 23–34, find f_x, f_y, and f_z.

23. $f(x, y, z) = 1 + xy^2 - 2z^2$

24. $f(x, y, z) = xy + yz + xz$

25. $f(x, y, z) = x - \sqrt{y^2 + z^2}$

26. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

27. $f(x, y, z) = \arcsin(xyz)$

28. $f(x, y, z) = \operatorname{arcsec}(x + yz)$

29. $f(x, y, z) = \ln(x + 2y + 3z)$

30. $f(x, y, z) = yz\ln(xy)$

31. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$

32. $f(x, y, z) = e^{-xyz}$

33. $f(x, y, z) = \tanh(x + 2y + 3z)$

34. $f(x, y, z) = \sinh(xy - z^2)$

In Exercises 35–40, find the partial derivative of the function with respect to each variable.

35. $f(t, \alpha) = \cos(2\pi t - \alpha)$

36. $g(u, v) = v^2 e^{(2u/v)}$

37. $h(\rho, \phi, \theta) = \rho\sin\phi\cos\theta$

38. $g(r, \theta, z) = r(1 - \cos\theta) - z$

39. Work done by the heart (Section 3.11, Exercise 59)

$$W(P, V, \delta, v, g) = PV + \frac{V\delta v^2}{2g}$$

40. Wilson lot size formula (Section 4.6, Exercise 61)

$$A(c, h, k, m, q) = \frac{km}{q} + cm + \frac{hq}{2}$$

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–54.

41. $f(x, y) = x + y + xy$

42. $f(x, y) = \sin xy$

43. $g(x, y) = x^2y + \cos y + y\sin x$

44. $h(x, y) = xe^y + y + 1$

45. $r(x, y) = \ln(x + y)$

46. $s(x, y) = \arctan(y/x)$

47. $w = x^2\tan(xy)$

48. $w = ye^{x^2-y}$

49. $w = x\sin(x^2y)$

50. $w = \dfrac{x - y}{x^2 + y}$

51. $f(x, y) = x^2y^3 - x^4 + y^5$

52. $g(x, y) = \cos x^2 - \sin 3y$

53. $z = x\sin(2x - y^2)$

54. $z = xe^{x/y^2}$

Mixed Partial Derivatives

In Exercises 55–60, verify that $w_{xy} = w_{yx}$.

55. $w = \ln(2x + 3y)$

56. $w = e^x + x\ln y + y\ln x$

57. $w = xy^2 + x^2y^3 + x^3y^4$

58. $w = x\sin y + y\sin x + xy$

59. $w = \dfrac{x^2}{y^3}$

60. $w = \dfrac{3x - y}{x + y}$

61. Which order of differentiation enables one to calculate f_{xy} faster: x first or y first? Try to answer without writing anything down.

a. $f(x, y) = x\sin y + e^y$

b. $f(x, y) = 1/x$

c. $f(x, y) = y + (x/y)$

d. $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$

e. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

f. $f(x, y) = x\ln xy$

62. The fifth-order partial derivative $\partial^5 f/\partial x^2\,\partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y? Try to answer without writing anything down.

a. $f(x, y) = y^2x^4e^x + 2$

b. $f(x, y) = y^2 + y(\sin x - x^4)$

c. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

d. $f(x, y) = xe^{y^2/2}$

Using the Partial Derivative Definition

In Exercises 63–66, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

63. $f(x, y) = 1 - x + y - 3x^2y, \quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(1, 2)$

64. $f(x, y) = 4 + 2x - 3y - xy^2, \quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 1)$

65. $f(x, y) = \sqrt{2x + 3y - 1}, \quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 3)$

66. $f(x, y) = \begin{cases} \dfrac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$

$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0, 0)$

67. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial z$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.

68. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial y$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial y$ at $(-1, 0, 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Differentiating Implicitly

69. Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

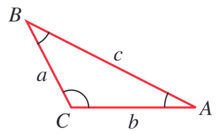
defines z as a function of the two independent variables x and y and the partial derivative exists.

70. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Exercises 71 and 72 are about the triangle shown here.



71. Express A implicitly as a function of a, b, and c and calculate $\partial A/\partial a$ and $\partial A/\partial b$.

72. Express a implicitly as a function of A, b, and B and calculate $\partial a/\partial A$ and $\partial a/\partial B$.

73. Two dependent variables Express v_x in terms of u and y if the equations $x = v \ln u$ and $y = u \ln v$ define u and v as functions of the independent variables x and y, and if v_x exists. (*Hint:* Differentiate both equations with respect to x and solve for v_x by eliminating u_x.)

74. Two dependent variables Find $\partial x/\partial u$ and $\partial y/\partial u$ if the equations $u = x^2 - y^2$ and $v = x^2 - y$ define x and y as functions of the independent variables u and v, and the partial derivatives exist. (See the hint in Exercise 73.) Then let $s = x^2 + y^2$ and find $\partial s/\partial u$.

Theory and Examples

75. Let $f(x, y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2, -1)$ and lying in **a.** the plane $x = 2$ **b.** the plane $y = -1$.

76. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in **a.** the plane $x = -1$ **b.** the plane $y = 1$.

In Exercises 77–80, find a function $z = f(x, y)$ whose partial derivatives are as given, or explain why this is impossible.

77. $\dfrac{\partial f}{\partial x} = 3x^2y^2 - 2x, \quad \dfrac{\partial f}{\partial y} = 2x^3y + 6y$

78. $\dfrac{\partial f}{\partial x} = 2xe^{xy^2} + x^2y^2e^{xy^2} + 3, \quad \dfrac{\partial f}{\partial y} = 2x^3ye^{xy^2} - e^y$

79. $\dfrac{\partial f}{\partial x} = \dfrac{2y}{(x + y)^2}, \quad \dfrac{\partial f}{\partial y} = \dfrac{2x}{(x + y)^2}$

80. $\dfrac{\partial f}{\partial x} = xy\cos(xy) + \sin(xy), \quad \dfrac{\partial f}{\partial y} = x\cos(xy)$

81. Let $f(x, y) = \begin{cases} y^3, & y \ge 0 \\ -y^2, & y < 0. \end{cases}$

Find f_x, f_y, f_{xy}, and f_{yx}, and state the domain for each partial derivative.

82. Let $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \ne 0, \\ 0, & \text{if } (x, y) = 0. \end{cases}$

a. Show that $\dfrac{\partial f}{\partial y}(x, 0) = x$ for all x, and $\dfrac{\partial f}{\partial x}(0, y) = -y$ for all y.

b. Show that $\dfrac{\partial^2 f}{\partial y\,\partial x}(0, 0) \ne \dfrac{\partial^2 f}{\partial x\,\partial y}(0, 0)$.

The **three-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is satisfied by steady-state temperature distributions $T = f(x, y, z)$ in space, by gravitational potentials, and by electrostatic potentials. The **two-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

obtained by dropping the $\partial^2 f/\partial z^2$ term from the previous equation, describes potentials and steady-state temperature distributions in a plane. The plane may be treated as a thin slice of the solid perpendicular to the z-axis.

Show that each function in Exercises 83–90 satisfies a Laplace equation.

83. $f(x, y, z) = x^2 + y^2 - 2z^2$

84. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

85. $f(x, y) = e^{-2y}\cos 2x$

86. $f(x, y) = \ln\sqrt{x^2 + y^2}$

87. $f(x, y) = 3x + 2y - 4$

88. $f(x, y) = \arctan\dfrac{x}{y}$

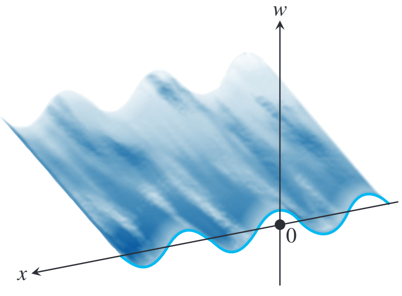
89. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

90. $f(x, y, z) = e^{3x+4y}\cos 5z$

The wave equation If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the **one-dimensional wave equation**

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial x^2},$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.



In our example, x is the distance across the ocean's surface, but in other applications, x might be the distance along a vibrating string, distance through air (sound waves), or distance through space (light waves). The number c varies with the medium and type of wave.

Show that the functions in Exercises 91–97 are all solutions of the wave equation.

91. $w = \sin(x + ct)$

92. $w = \cos(2x + 2ct)$

93. $w = \sin(x + ct) + \cos(2x + 2ct)$

94. $w = \ln(2x + 2ct)$

95. $w = \tan(2x - 2ct)$

96. $w = 5\cos(3x + 3ct) + e^{x+ct}$

97. $w = f(u)$, where f is a differentiable function of u, and $u = a(x + ct)$, where a is a constant

98. Does a function $f(x, y)$ with continuous first partial derivatives throughout an open region R have to be continuous on R? Give reasons for your answer.

99. If a function $f(x, y)$ has continuous second partial derivatives throughout an open region R, must the first-order partial derivatives of f be continuous on R? Give reasons for your answer.

100. The heat equation An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

Show that $u(x, t) = \sin(\alpha x) \cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

101. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$. (*Hint:* Use Theorem 4 and show that f is not continuous at $(0, 0)$.)

102. Let $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

103. The Korteweg–de Vries equation

This nonlinear differential equation, which describes wave motion on shallow water surfaces, is given by

$$u_t + u_{xxx} + 12uu_x = 0.$$

Show that $u(x, t) = \operatorname{sech}^2(x - t)$ satisfies the Korteweg–de Vries equation.

104. Show that $T = \dfrac{1}{\sqrt{x^2 + y^2}}$ satisfies the equation $T_{xx} + T_{yy} = T^3$.

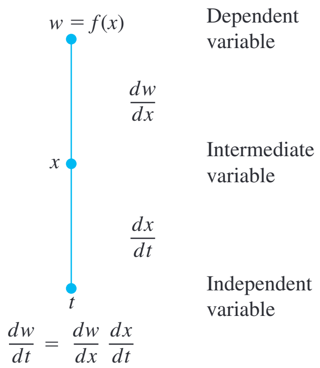
14.4 The Chain Rule

The Chain Rule for functions of a single variable studied in Section 3.6 says that if $w = f(x)$ is a differentiable function of x, and $x = g(t)$ is a differentiable function of t, then w is a differentiable function of t, and dw/dt can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

To find dw/dt, we read down the route from w to t, multiplying derivatives along the way.

Chain Rule



For this composite function $w(t) = f(g(t))$, we can think of t as the independent variable and $x = g(t)$ as the "intermediate variable" because t determines the value of x that in turn gives the value of w from the function f. We display the Chain Rule in a "dependency diagram" in the margin. Such diagrams capture which variables depend on which.

For functions of several variables the Chain Rule has more than one form, which depends on how many independent and intermediate variables are involved. However, once the variables are taken into account, the Chain Rule works in the same way we just discussed.

Functions of Two Variables

The Chain Rule formula for a differentiable function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

THEOREM 5—Chain Rule for Functions of One Independent Variable and Two Intermediate Variables
If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composition $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$