62. $f(x, y, z) = \ln(x^2 + y + z^2), \quad (-1, 2, 1)$

63. $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad (1, -1, \sqrt{2})$

64. $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}, \quad (1, 0, -2)$

In Exercises 65–68, find and sketch the domain of f. Then find an equation for the level curve or surface of the function passing through the given point.

65. $f(x, y) = \sum_{n=0}^{\infty} \left(\dfrac{x}{y}\right)^n, \quad (1, 2)$

66. $g(x, y, z) = \sum_{n=0}^{\infty} \dfrac{(x + y)^n}{n! z^n}, \quad (\ln 4, \ln 9, 2)$

67. $f(x, y) = \int_x^y \dfrac{d\theta}{\sqrt{1 - \theta^2}}, \quad (0, 1)$

68. $g(x, y, z) = \int_x^y \dfrac{dt}{1 + t^2} + \int_0^z \dfrac{d\theta}{\sqrt{4 - \theta^2}}, \quad (0, 1, \sqrt{3})$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for each of the functions in Exercises 69–72.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of f through the given point.

69. $f(x, y) = x \sin\dfrac{y}{2} + y \sin 2x, \quad 0 \le x \le 5\pi, \quad 0 \le y \le 5\pi,$ $P(3\pi, 3\pi)$

70. $f(x, y) = (\sin x)(\cos y)e^{\sqrt{x^2+y^2}/8}, \quad 0 \le x \le 5\pi,$ $0 \le y \le 5\pi, \quad P(4\pi, 4\pi)$

71. $f(x, y) = \sin(x + 2\cos y), \quad -2\pi \le x \le 2\pi,$ $-2\pi \le y \le 2\pi, \quad P(\pi, \pi)$

72. $f(x, y) = e^{(x^{0.1} - y)} \sin(x^2 + y^2), \quad 0 \le x \le 2\pi,$ $-2\pi \le y \le \pi, \quad P(\pi, -\pi)$

Use a CAS to plot the implicitly defined level surfaces in Exercises 73–76.

73. $4 \ln(x^2 + y^2 + z^2) = 1$ **74.** $x^2 + z^2 = 1$

75. $x + y^2 - 3z^2 = 1$

76. $\sin\left(\dfrac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2$

Parametrized Surfaces Just as you describe curves in the plane parametrically with a pair of equations $x = f(t)$, $y = g(t)$ defined on some parameter interval I, you can sometimes describe surfaces in space with a triple of equations $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ defined on some parameter rectangle $a \le u \le b$, $c \le v \le d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode*. (Parametrized surfaces are discussed in detail in Section 16.5.) Use a CAS to plot the surfaces in Exercises 77–80. Also plot several level curves in the xy-plane.

77. $x = u \cos v, \quad y = u \sin v, \quad z = u, \quad 0 \le u \le 2,$ $0 \le v \le 2\pi$

78. $x = u \cos v, \quad y = u \sin v, \quad z = v, \quad 0 \le u \le 2,$ $0 \le v \le 2\pi$

79. $x = (2 + \cos u)\cos v, \quad y = (2 + \cos u)\sin v, \quad z = \sin u,$ $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$

80. $x = 2\cos u \cos v, \quad y = 2\cos u \sin v, \quad z = 2\sin u,$ $0 \le u \le 2\pi, \quad 0 \le v \le \pi$

14.2 Limits and Continuity in Higher Dimensions

In this section we develop limits and continuity for multivariable functions. The theory is similar to that developed for single-variable functions, but since we now have more than one independent variable, there is additional complexity that requires some new ideas.

Limits for Functions of Two Variables

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that when (x_0, y_0) lies in the interior of f's domain, (x, y) can approach (x_0, y_0) from any direction, not just from the left or the right. For the limit to exist, the same limiting value must be obtained whatever direction of approach is taken. We illustrate this issue in several examples following the definition.

DEFINITION Suppose that every open circular disk centered at (x_0, y_0) contains a point in the domain of f other than (x_0, y_0) itself. We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

The definition of limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is made sufficiently small (but not 0). The definition applies to interior points (x_0, y_0) as well as boundary points of the domain of f, although a boundary point need not lie within the domain. The points (x, y) that approach (x_0, y_0) are always taken to be in the domain of f. See Figure 14.12.

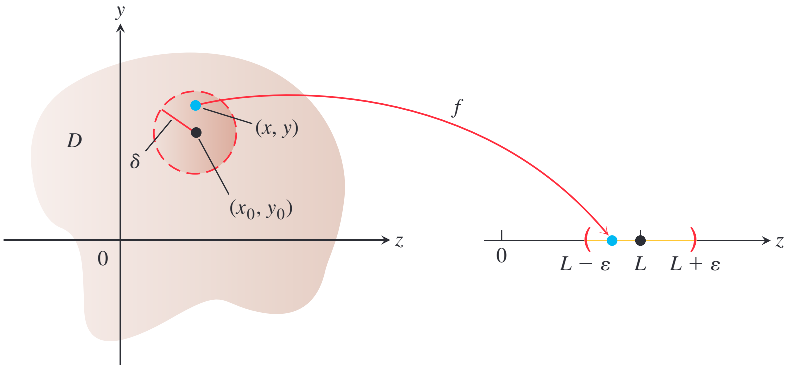


FIGURE 14.12 In the limit definition, δ is the radius of a disk centered at (x_0, y_0). For all points (x, y) within this disk, the function values $f(x, y)$ lie inside the corresponding interval $(L - \varepsilon, L + \varepsilon)$.

As for functions of a single variable, it can be shown that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0 \tag{1}$$

$$\lim_{(x,y)\to(x_0,y_0)} y = y_0 \tag{2}$$

$$\lim_{(x,y)\to(x_0,y_0)} k = k \qquad \text{(any number } k\text{)}. \tag{3}$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\varepsilon > 0$ is chosen. If we let δ equal this ε, we see that if

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \varepsilon,$$

then

$$\begin{aligned} \sqrt{(x - x_0)^2} &< \varepsilon && (x - x_0)^2 \le (x - x_0)^2 + (y - y_0)^2 \\ |x - x_0| &< \varepsilon && \sqrt{a^2} = |a| \\ |f(x, y) - x_0| &< \varepsilon. && x = f(x, y) \end{aligned}$$

That is,

$$|f(x, y) - x_0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So a δ has been found satisfying the requirement of the definition, and therefore we have proved that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \lim_{(x,y)\to(x_0,y_0)} x = x_0.$$

Equation (1) is a special case of the more general formula

$$\lim_{(x,y)\to(x_0,y_0)} g(x) = \lim_{x\to x_0} g(x), \tag{4}$$

according to which, if $f(x, y)$ can be expressed as a function g of a single variable x, then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ depends only on what happens to g as x approaches x_0. Similarly, the following formula generalizes Equation (2):

$$\lim_{(x,y)\to(x_0,y_0)} h(y) = \lim_{y\to y_0} h(y) \tag{5}$$

As with single-variable functions, the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, constant multiples, products, quotients, powers, and roots. These facts are summarized in Theorem 1.

THEOREM 1—Properties of Limits of Functions of Two Variables
The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) - g(x,y)] = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL \quad$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,
   n a positive integer, and if n is even, we assume that $L > 0$.
8. *Composition Rule:* If $h(z)$ is continuous at $z = L$, then
   $\lim_{(x,y)\to(x_0,y_0)} h(f(x,y)) = h(L)$.

Although we will not prove Theorem 1 here, we give an informal discussion of why it is true. If (x, y) is sufficiently close to (x_0, y_0), then $f(x, y)$ is close to L and $g(x, y)$ is close to M (from the informal interpretation of limits). It is then reasonable that $f(x, y) + g(x, y)$ is close to $L + M$; $f(x, y) - g(x, y)$ is close to $L - M$; $kf(x, y)$ is close to kL; $f(x, y)g(x, y)$ is close to LM; and $f(x, y)/g(x, y)$ is close to L/M if $M \neq 0$. Similarly, powers and roots of f are close to those of L, and a continuous function h composed with f has a value close to its value $h(L)$ when applied to L.

When we apply Theorem 1 and Equations (1)–(3) to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \to (x_0, y_0)$ can be calculated by evaluating the functions at (x_0, y_0). The only requirement is that the rational functions be defined at (x_0, y_0).

EXAMPLE 1 In this example, we combine Equations (1)–(5) with the results in Theorem 1 to calculate the limits.

(a)
$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b)
$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{\lim_{(x,y)\to(3,-4)} (x^2 + y^2)} \qquad \text{Rule 7}$$
$$= \sqrt{3^2 + (-4)^2} \qquad \text{Rules 1 and 6 and Eq. (1) and (2)}$$
$$= \sqrt{25} = 5$$

(c)
$$\lim_{(x,y)\to(\pi/2,0)} \left(\frac{x}{\sin x} - \frac{\sin y}{y} \right) = \lim_{(x,y)\to(\pi/2,0)} \frac{x}{\sin x} - \lim_{(x,y)\to(\pi/2,0)} \frac{\sin y}{y} \qquad \text{Rule 2}$$
$$= \lim_{x\to\pi/2} \frac{x}{\sin x} - \lim_{y\to 0} \frac{\sin y}{y} \qquad \text{Eq. (4) and (5)}$$
$$= \frac{\pi}{2} - 1 \qquad \text{Theorem 6, Section 2.4}$$

EXAMPLE 2 Find $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} && \text{Multiply by a form equal to 1.} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\
&= \left(\lim_{(x,y)\to(0,0)} x\right)\left[\left(\lim_{(x,y)\to(0,0)} \sqrt{x}\right) + \left(\lim_{(x,y)\to(0,0)} \sqrt{y}\right)\right] && \text{Rules 4 and 1} \\
&= \left(\lim_{(x,y)\to(0,0)} x\right)\left[\sqrt{\lim_{(x,y)\to(0,0)} x} + \sqrt{\lim_{(x,y)\to(0,0)} y}\right] && \text{Rule 7} \\
&= (0)\left[\sqrt{0} + \sqrt{0}\right] = 0 && \text{Eq. (1) and (2)}
\end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (where we would have $x - y = 0$) is *not* in the domain of the function

$$f(x, y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

■

EXAMPLE 3 Find $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2 + y^2}$ if it exists.

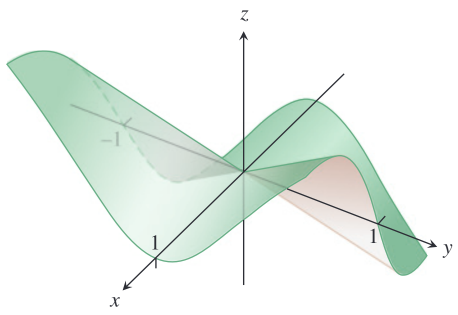


FIGURE 14.13 The surface graph suggests that the limit of the function in Example 3 must be 0, if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0 (see Figure 14.13). To see whether this is true, we apply the definition of limit.

Let $\varepsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left|\frac{4xy^2}{x^2 + y^2} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$, we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}. \qquad \frac{y^2}{x^2 + y^2} \leq 1$$

So if we choose $\delta = \varepsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left|\frac{4xy^2}{x^2 + y^2} - 0\right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

It follows from the definition that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

■

EXAMPLE 4 If $f(x, y) = \dfrac{y}{x}$, does $\lim_{(x, y) \to (0,0)} f(x, y)$ exist?

Solution The domain of f does not include the y-axis, so we do not consider any points (x, y) where $x = 0$ in the approach toward the origin $(0, 0)$. Along the x-axis, the value of the function is $f(x, 0) = 0$ for all $x \neq 0$. So if the limit does exist as $(x, y) \to (0, 0)$, the value of the limit must be $L = 0$. On the other hand, along the line $y = x$, the value of the function is $f(x, x) = x/x = 1$ for all $x \neq 0$. That is, the function f approaches the value 1 along the line $y = x$. This means that for every disk of radius δ centered at $(0, 0)$, the disk will contain points $(x, 0)$ on the x-axis where the value of the function is 0, and also points (x, x) along the line $y = x$ where the value of the function is 1. So no matter how small we choose δ as the radius of the disk in Figure 14.12, there will be points within the disk for which the function values differ by 1. Therefore, the limit cannot exist because we can take ε to be any number less than 1 in the limit definition and deny that $L = 0$ or 1, or any other real number. The limit does not exist because we have different limiting values along different paths approaching the point $(0, 0)$. ■

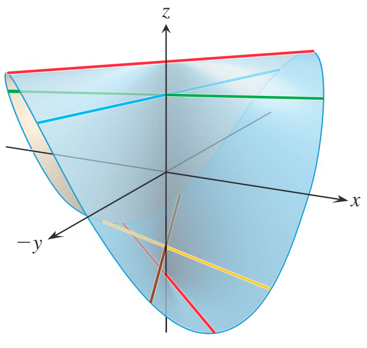
Continuity

As with functions of a single variable, continuity is defined in terms of limits.

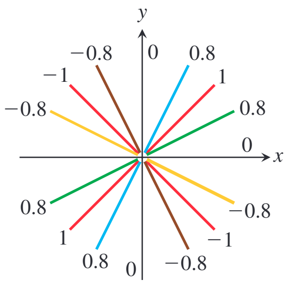
DEFINITION Suppose that every open circular disk centered at (x_0, y_0) contains a point in the domain of f other than (x_0, y_0) itself. Then a function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ exists, and
3. $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.



(a)



(b)

FIGURE 14.14 (a) The graph of

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

The function is continuous at every point except the origin. (b) The value of f along each line $y = mx$, $x \neq 0$, is constant but varies with m (Example 5).

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f.

A consequence of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

Solution The function f is continuous at every point (x, y) except $(0, 0)$ because its values at points other than $(0, 0)$ are given by a rational function of x and y, and therefore at those points the limiting value is simply obtained by substituting the values of x and y into that rational expression.

At $(0, 0)$, the value of f is defined, but f has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m, the function f has a constant value on the "punctured" line $y = mx, x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with each value of the slope m. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous at the origin. ■

Examples 4 and 5 illustrate an important point about limits of functions of two or more variables. For a limit to exist at a point, the limit must be the same along every approach path. This result is analogous to the single-variable case where both the left- and right-sided limits had to have the same value. For functions of two or more variables, if we ever find paths with different limits, we know the function has no limit at the point they approach.

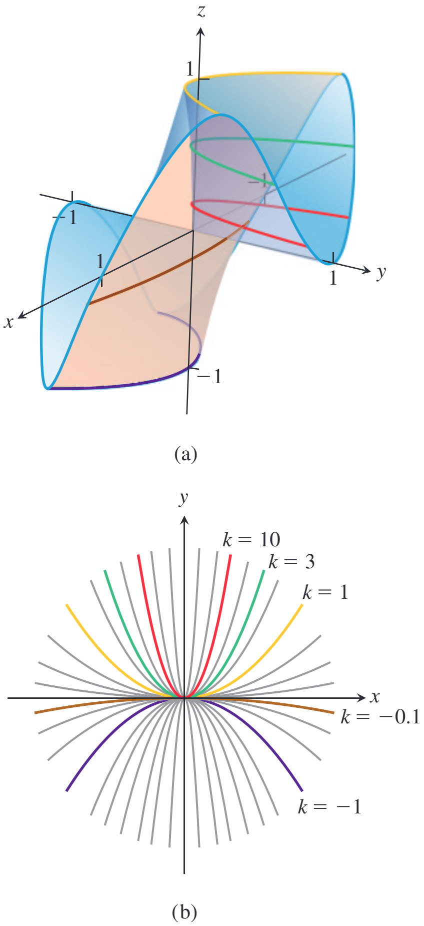


FIGURE 14.15 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. (b) Along each path $y = kx^2$, $x \neq 0$, the value of f is constant, but varies with k (Example 6).

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as (x, y) approaches $(0, 0)$.

Solution As (x, y) approaches $(0, 0)$, both the numerator and the denominator approach 0, which gives the indeterminate form $0/0$. We examine the values of f along parabolic curves that end at $(0, 0)$. Along the curve $y = kx^2, x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$. ■

It can be shown that the function in Example 6 has limit 0 along every straight line path $y = mx$ (Exercise 57). This implies the following observation:

Having the same limit along all straight lines approaching (x_0, y_0) does not imply that a limit exists at (x_0, y_0).

Whenever it is correctly defined, the composition of continuous functions is also continuous. The only requirement is that each function be continuous where it is applied. The proof, omitted here, is similar to that for functions of a single variable (Theorem 9 in Section 2.5).

Continuity of Compositions

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composition $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos\frac{xy}{x^2+1}, \qquad \ln(1 + x^2y^2)$$

are continuous at every point (x, y).

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and compositions all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \to (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

Extreme Values of Continuous Functions on Closed, Bounded Sets

The Extreme Value Theorem (Theorem 1, Section 4.1) states that a function of a single variable that is continuous at every point of a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$. The same holds true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R in the plane (like a line segment, a disk, or a filled-in triangle). The function takes on an absolute maximum value at some point in R and an absolute minimum value at some point in R. The function may take on a maximum or minimum value more than once over R.

Similar results hold for functions of three or more variables. A continuous function $w = f(x, y, z)$ must take on absolute maximum and minimum values on any closed, bounded set (such as a solid ball or cube, spherical shell, or rectangular solid) on which it is defined. We will learn how to find these extreme values in Section 14.7.

EXERCISES 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\displaystyle \lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

3. $\displaystyle \lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\displaystyle \lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$

5. $\displaystyle \lim_{(x,y)\to(0,\pi/4)} \sec x \tan y$

6. $\displaystyle \lim_{(x,y)\to(0,0)} \cos\frac{x^2 + y^3}{x + y + 1}$

7. $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$

8. $\lim_{(x,y)\to(1,1)} \ln\left|1 + x^2y^2\right|$

9. $\lim_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

10. $\lim_{(x,y)\to(1/27,\pi^3)} \cos\sqrt[3]{xy}$

11. $\lim_{(x,y)\to(1,\pi/6)} \dfrac{x \sin y}{x^2 + 1}$

12. $\lim_{(x,y)\to(\pi/2,0)} \dfrac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

14. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

15. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy - y - 2x + 2}{x - 1}$

16. $\lim_{\substack{(x,y)\to(2,-4)\\ x\neq -4,\, x\neq x^2}} \dfrac{y + 4}{x^2y - xy + 4x^2 - 4x}$

17. $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \dfrac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

18. $\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x + y - 4}{\sqrt{x + y} - 2}$

19. $\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \dfrac{\sqrt{2x - y} - 2}{2x - y - 4}$

20. $\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}$

21. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

22. $\lim_{(x,y)\to(0,0)} \dfrac{1 - \cos(xy)}{xy}$

23. $\lim_{(x,y)\to(1,-1)} \dfrac{x^3 + y^3}{x + y}$

24. $\lim_{(x,y)\to(2,2)} \dfrac{x - y}{x^4 - y^4}$

Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim_{P\to(1,3,4)} \left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$

26. $\lim_{P\to(1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$

27. $\lim_{P\to(\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

28. $\lim_{P\to(-1/4,\pi/2,2)} \tan^{-1} xyz$

29. $\lim_{P\to(\pi,0,3)} ze^{-2y}\cos 2x$

30. $\lim_{P\to(2,-3,6)} \ln\sqrt{x^2 + y^2 + z^2}$

Continuity for Two Variables

At what points (x, y) in the plane are the functions in Exercises 31–34 continuous?

31. a. $f(x, y) = \sin(x + y)$ **b.** $f(x, y) = \ln(x^2 + y^2)$

32. a. $f(x, y) = \dfrac{x + y}{x - y}$ **b.** $f(x, y) = \dfrac{y}{x^2 + 1}$

33. a. $g(x, y) = \sin\dfrac{1}{xy}$ **b.** $g(x, y) = \dfrac{x + y}{2 + \cos x}$

34. a. $g(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$ **b.** $g(x, y) = \dfrac{1}{x^2 - y}$

Continuity for Three Variables

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

35. a. $f(x, y, z) = x^2 + y^2 - 2z^2$

b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$

36. a. $f(x, y, z) = \ln xyz$ **b.** $f(x, y, z) = e^{x+y}\cos z$

37. a. $h(x, y, z) = xy\sin\dfrac{1}{z}$ **b.** $h(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$

38. a. $h(x, y, z) = \dfrac{1}{|y| + |z|}$ **b.** $h(x, y, z) = \dfrac{1}{|xy| + |z|}$

39. a. $h(x, y, z) = \ln(z - x^2 - y^2 - 1)$

b. $h(x, y, z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

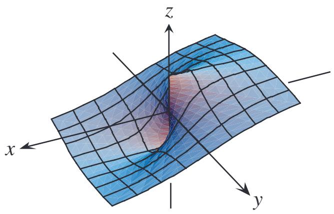
40. a. $h(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$

b. $h(x, y, z) = \dfrac{1}{4 - \sqrt{x^2 + y^2 + z^2 - 9}}$

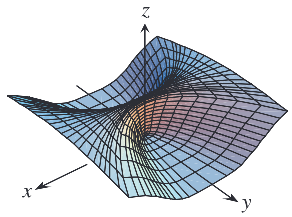
No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$



42. $f(x, y) = \dfrac{x^4}{x^4 + y^2}$



43. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$

44. $f(x, y) = \dfrac{xy}{|xy|}$

45. $g(x, y) = \dfrac{x - y}{x + y}$

46. $g(x, y) = \dfrac{x^2 - y}{x - y}$

47. $h(x, y) = \dfrac{x^2 + y}{y}$

48. $h(x, y) = \dfrac{x^2y}{x^4 + y^2}$

Theory and Examples

In Exercises 49–54, show that the limits do not exist.

49. $\lim_{(x,y)\to(1,1)} \dfrac{xy^2 - 1}{y - 1}$

50. $\lim_{(x,y)\to(1,-1)} \dfrac{xy + 1}{x^2 - y^2}$

51. $\lim_{(x,y)\to(0,1)} \dfrac{x \ln y}{x^2 + (\ln y)^2}$

52. $\lim_{(x,y)\to(1,0)} \dfrac{xe^y - 1}{xe^y - 1 + y}$

53. $\lim_{(x,y)\to(0,0)} \dfrac{y + \sin x}{x + \sin y}$

54. $\lim_{(x,y)\to(1,1)} \dfrac{\tan y - y\tan x}{y - x}$

55. Let $f(x, y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

a. $\lim_{(x,y)\to(0,1)} f(x, y)$

b. $\lim_{(x,y)\to(2,3)} f(x, y)$

c. $\lim_{(x,y)\to(0,0)} f(x, y)$

56. Let $f(x, y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}.$

Find the following limits.

a. $\lim_{(x,y)\to(3,-2)} f(x, y)$

b. $\lim_{(x,y)\to(-2,1)} f(x, y)$

c. $\lim_{(x,y)\to(0,0)} f(x, y)$

57. Show that the function in Example 6 has limit 0 along every straight line approaching $(0, 0)$.

58. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

if f is continuous at (x_0, y_0)? If f is not continuous at (x_0, y_0)? Give reasons for your answers.

The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Use this result to support your answers to the questions in Exercises 59–62.

59. Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

60. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

61. Does knowing that $|\sin(1/x)| \leq 1$ tell you anything about

$$\lim_{(x,y)\to(0,0)} y \sin\frac{1}{x}?$$

Give reasons for your answer.

62. Does knowing that $|\cos(1/y)| \leq 1$ tell you anything about

$$\lim_{(x,y)\to(0,0)} x \cos\frac{1}{y}?$$

Give reasons for your answer.

63. (*Continuation of Example 5.*)

a. Reread Example 5. Then substitute $m = \tan\theta$ into the formula

$$f(x, y)\Big|_{y=mx} = \frac{2m}{1 + m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b. Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \to (0, 0)$ along the line $y = mx$ varies from -1 to 1, depending on the angle of approach.

64. Continuous extension Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing Variables to Polar Coordinates

If you cannot make any headway with $\lim_{(x,y)\to(0,0)} f(x, y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r\cos\theta$, $y = r\sin\theta$, and investigate the limit of the resulting expression as $r \to 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |f(r, \theta) - L| < \varepsilon. \tag{1}$$

If such an L exists, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L.$$

For instance,

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0.$$

To verify the last of these equalities, we need to show that Equation (1) is satisfied with $f(r, \theta) = r\cos^3\theta$ and $L = 0$. That is, we need to show that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |r\cos^3\theta - 0| < \varepsilon.$$

Since

$$|r\cos^3\theta| = |r||\cos^3\theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \varepsilon$.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x,y)\to(0,0)} x^2/(x^2+y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \to 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. Example 5 illustrates this point. In polar coordinates, $f(x, y) = (2x^2y)/(x^4 + y^2)$ becomes

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + \sin^2\theta}$$

for $r \neq 0$. If we hold θ constant and let $r \to 0$, the limit is 0. On the path $y = x^2$, however, we have $r\sin\theta = r^2\cos^2\theta$ and

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + (r\cos^2\theta)^2} \\ &= \frac{2r\cos^2\theta\sin\theta}{2r^2\cos^4\theta} = \frac{r\sin\theta}{r^2\cos^2\theta} = 1. \end{aligned}$$

In Exercises 65–70, find the limit of f as $(x, y) \to (0, 0)$ or show that the limit does not exist.

65. $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$

66. $f(x, y) = \cos\left(\dfrac{x^3 - y^3}{x^2 + y^2}\right)$

67. $f(x, y) = \dfrac{y^2}{x^2 + y^2}$

68. $f(x, y) = \dfrac{2x}{x^2 + x + y^2}$

69. $f(x, y) = \tan^{-1}\left(\dfrac{|x| + |y|}{x^2 + y^2}\right)$

70. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 71 and 72, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

71. $f(x, y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

72. $f(x, y) = \dfrac{3x^2y}{x^2 + y^2}$

Using the Limit Definition

Each of Exercises 73–78 gives a function $f(x, y)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y),

$$\sqrt{x^2 + y^2} < \delta \quad\Rightarrow\quad |f(x, y) - f(0, 0)| < \varepsilon.$$

73. $f(x, y) = x^2 + y^2, \quad \varepsilon = 0.01$

74. $f(x, y) = y/(x^2 + 1), \quad \varepsilon = 0.05$

75. $f(x, y) = (x + y)/(x^2 + 1), \quad \varepsilon = 0.01$

76. $f(x, y) = (x + y)/(2 + \cos x), \quad \varepsilon = 0.02$

77. $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \varepsilon = 0.04$

78. $f(x, y) = \dfrac{x^3 + y^4}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \varepsilon = 0.02$

Each of Exercises 79–82 gives a function $f(x, y, z)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z),

$$\sqrt{x^2 + y^2 + z^2} < \delta \quad\Rightarrow\quad |f(x, y, z) - f(0, 0, 0)| < \varepsilon.$$

79. $f(x, y, z) = x^2 + y^2 + z^2, \quad \varepsilon = 0.015$

80. $f(x, y, z) = xyz, \quad \varepsilon = 0.008$

81. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \varepsilon = 0.015$

82. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \varepsilon = 0.03$

83. Show that $f(x, y, z) = x + y - z$ is continuous at every point (x_0, y_0, z_0).

84. Show that $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.

14.3 Partial Derivatives

The calculus of several variables is similar to single-variable calculus applied to several variables, one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the familiar rules for differentiating functions of a single variable. The idea of *differentiability* for functions of several variables requires more than the existence of the partial derivatives, because a point can be approached from many different directions. However, we will see that differentiable functions of several variables behave similarly to differentiable single-variable functions. In particular, they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.16). This curve is the graph