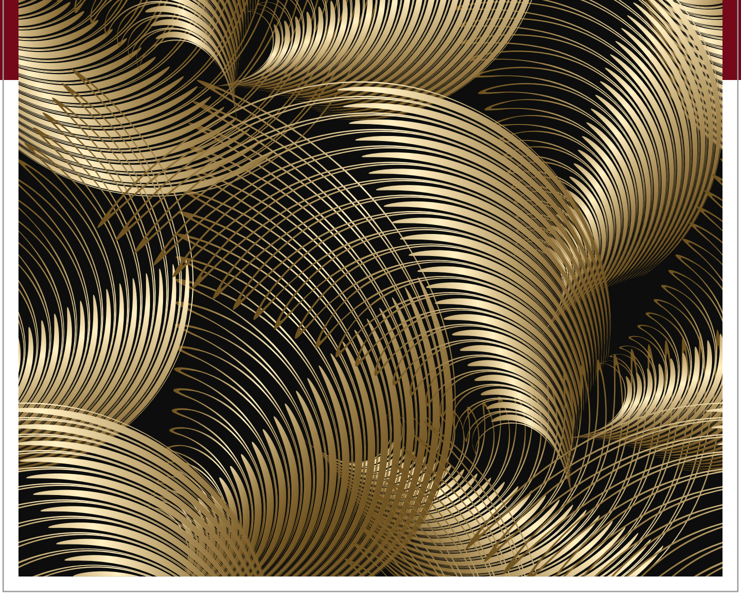
# 13

# Vector-Valued Functions and Motion in Space



OVERVIEW In this chapter we introduce the calculus of vector-valued functions. The domains of these functions are sets of real numbers, as before, but their ranges consist of vectors instead of scalars. When a vector-valued function changes, the change can occur in both magnitude and direction, so the derivative is itself a vector. The integral of a vector-valued function is also a vector. We use the calculus of these functions to describe the paths and motions of objects moving in a plane or in space, so their velocities and accelerations are given by vectors.

## 13.1 Curves in Space and Their Tangents

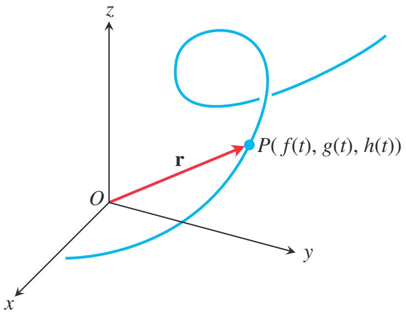



**FIGURE 13.1** The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

When a particle moves through space during a time interval $I$, we think of the particle's coordinates as functions defined on $I$:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I. \tag{1}$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equations and interval in Equation (1) parametrize the curve.

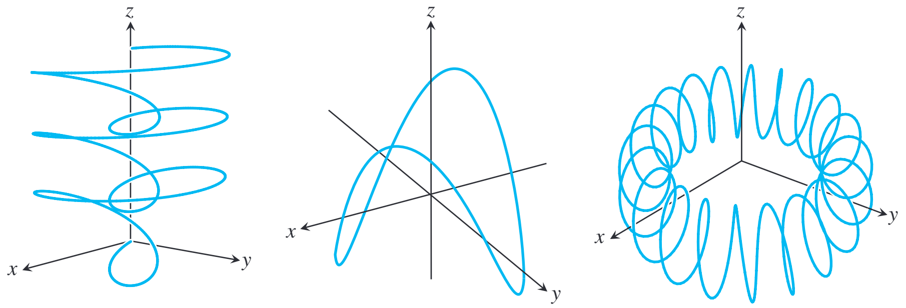
A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \tag{2}$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time $t$ is the particle's position vector (Figure 13.1). The functions $f$, $g$, and $h$ are the **component functions** (or components) of the position vector. We think of the particle's path as the curve traced by $\mathbf{r}$ during the time interval $I$. Figure 13.2 displays several space curves generated by a computer graphing program.

Equation (2) defines $\mathbf{r}$ as a vector function of the real variable $t$ on the interval $I$. More generally, a **vector-valued function** or **vector function** on a domain set $D$ is a rule that assigns a vector in space to each element in $D$. For now, the domains will be intervals of real numbers, and the graph of the function represents a curve in space. Vector functions on a domain in the plane or in space give rise to "vector fields," which are important to the study of fluid flows, gravitational fields, and electromagnetic phenomena. We investigate vector fields and their applications in Chapter 16.

Real-valued functions are often called **scalar functions** to distinguish them from vector functions. The components of $\mathbf{r}$ in Equation (2) are scalar functions of $t$. The domain of a vector-valued function is the common domain of its components.




$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$

(a)

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$

(b)

$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$

(c)

**FIGURE 13.2** Space curves are defined by the position vectors $\mathbf{r}(t)$.

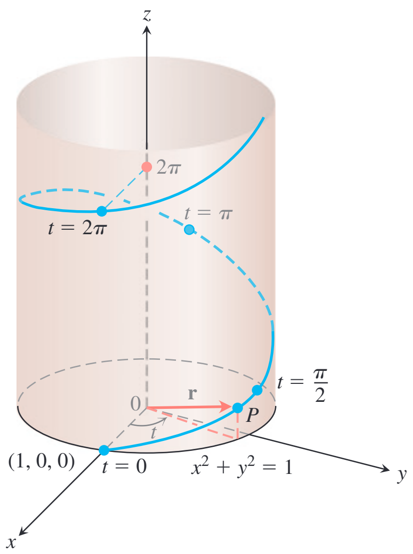



**FIGURE 13.3** The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).

**EXAMPLE 1** Graph the vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

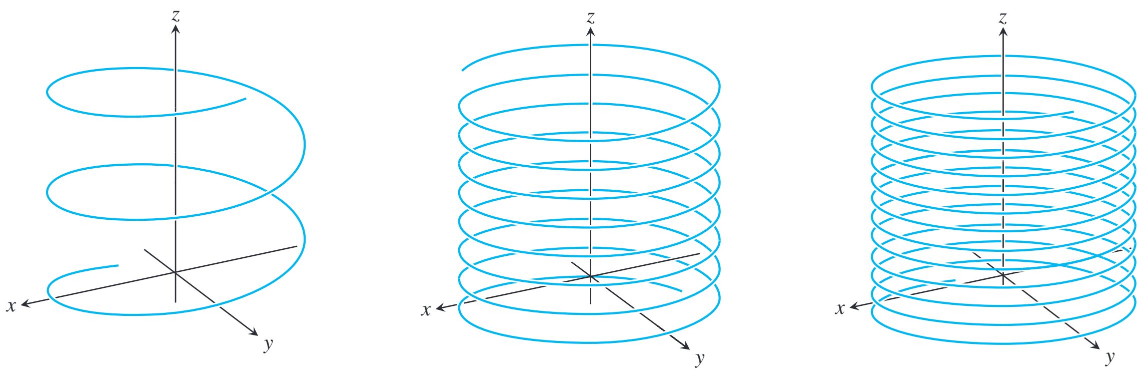
**Solution** This vector function $\mathbf{r}(t)$ is defined for all real values of $t$. The curve traced by $\mathbf{r}$ winds around the circular cylinder $x^2 + y^2 = 1$ (Figure 13.3). The curve lies on the cylinder because the **i**- and **j**-components of $\mathbf{r}$, being the $x$- and $y$-coordinates of the tip of $\mathbf{r}$, satisfy the cylinder's equation:

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

The curve rises as the **k**-component $z = t$ increases. Each time $t$ increases by $2\pi$, the curve completes one turn around the cylinder. The curve is called a **helix** (from an ancient Greek word for "spiral"). The equations

$$x = \cos t, \qquad y = \sin t, \qquad z = t$$

parametrize the helix. The domain is the largest set of points $t$ for which all three equations are defined, or $-\infty < t < \infty$ for this example. Figure 13.4 shows more helices. ■




$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\qquad$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\qquad$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

**FIGURE 13.4** Helices spiral upward around a cylinder, like coiled springs.

## Limits and Continuity

The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

**DEFINITION** Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain $D$, and let $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has **limit L** as $t$ approaches $t_0$ and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that, for all $t \in D$,

$$|\mathbf{r}(t) - \mathbf{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

If $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, then it can be shown that $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$ precisely when

$$\lim_{t \to t_0} f(t) = L_1, \qquad \lim_{t \to t_0} g(t) = L_2, \qquad \text{and} \qquad \lim_{t \to t_0} h(t) = L_3.$$

We omit the proof. The equation

To calculate the limit of a vector function, we find the limit of each component scalar function.

$$\lim_{t \to t_0} \mathbf{r}(t) = \left(\lim_{t \to t_0} f(t)\right)\mathbf{i} + \left(\lim_{t \to t_0} g(t)\right)\mathbf{j} + \left(\lim_{t \to t_0} h(t)\right)\mathbf{k} \tag{3}$$

provides a practical way to calculate limits of vector functions.

**EXAMPLE 2** If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned} \lim_{t \to \pi/4} \mathbf{r}(t) &= \left(\lim_{t \to \pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t \to \pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t \to \pi/4} t\right)\mathbf{k} \\ &= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}. \end{aligned}$$

We define continuity for vector functions the same way we define continuity for scalar functions defined over an interval.

**DEFINITION** A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

From Equation (3), we see that $\mathbf{r}(t)$ is continuous at $t = t_0$ if and only if each component function is continuous there (Exercise 45).

**EXAMPLE 3**

**(a)** All the space curves shown in Figures 13.2 and 13.4 are continuous because their component functions are continuous at every value of $t$ in $(-\infty, \infty)$.

**(b)** The function

$$\mathbf{g}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \lfloor t \rfloor\mathbf{k}$$

is discontinuous at every integer, because the greatest integer function $\lfloor t \rfloor$ is discontinuous at every integer.

## Derivatives and Motion

Suppose that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is the position vector of a particle moving along a curve in space and that $f$, $g$, and $h$ are differentiable functions of $t$. Then the difference between the particle's positions at time $t$ and time $t + \Delta t$ is the vector

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

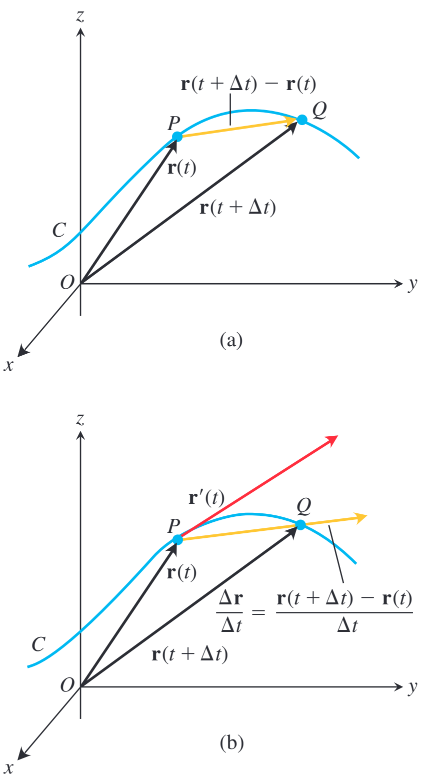



**FIGURE 13.5** As $\Delta t \to 0$, the point $Q$ approaches the point $P$ along the curve $C$. In the limit, the vector $\overrightarrow{PQ}/\Delta t$ becomes the tangent vector $\mathbf{r}'(t)$.

(Figure 13.5a). In terms of components,

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}(t+\Delta t) - \mathbf{r}(t)\\ &= [f(t+\Delta t)\mathbf{i} + g(t+\Delta t)\mathbf{j} + h(t+\Delta t)\mathbf{k}] - [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]\\ &= [f(t+\Delta t) - f(t)]\mathbf{i} + [g(t+\Delta t) - g(t)]\mathbf{j} + [h(t+\Delta t) - h(t)]\mathbf{k}.\end{aligned}$$

As $\Delta t$ approaches zero, three things seem to happen simultaneously. First, $Q$ approaches $P$ along the curve. Second, the secant line $PQ$ seems to approach a limiting position tangent to the curve at $P$. Third, the quotient $\Delta\mathbf{r}/\Delta t$ (Figure 13.5b) approaches the limit

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} &= \left[\lim_{\Delta t\to 0}\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]\mathbf{i} + \left[\lim_{\Delta t\to 0}\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]\mathbf{j}\\ &\quad + \left[\lim_{\Delta t\to 0}\frac{h(t+\Delta t) - h(t)}{\Delta t}\right]\mathbf{k}\\ &= \left[\frac{df}{dt}\right]\mathbf{i} + \left[\frac{dg}{dt}\right]\mathbf{j} + \left[\frac{dh}{dt}\right]\mathbf{k}.\end{aligned}$$

These observations lead us to the following definition.

**DEFINITION** The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at $t$** if $f$, $g$, and $h$ have derivatives at $t$. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

A vector function $\mathbf{r}$ is **differentiable** if it is differentiable at every point of its domain.

The geometric significance of the definition of derivative is shown in Figure 13.5. The points $P$ and $Q$ have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+\Delta t)$, and the vector $\overrightarrow{PQ}$ is represented by $\mathbf{r}(t+\Delta t) - \mathbf{r}(t)$. For $\Delta t > 0$, the scalar multiple $(1/\Delta t)(\mathbf{r}(t+\Delta t) - \mathbf{r}(t))$ points in the same direction as the vector $\overrightarrow{PQ}$. As $\Delta t \to 0$, this vector approaches the vector $\mathbf{r}'(t)$, which is a vector tangent to the curve at $P$, as long as it is different from the zero vector $\mathbf{0}$ (Figure 13.5b).

The curve traced by $\mathbf{r}$ is **smooth** if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$, that is, if $f$, $g$, and $h$ have continuous first derivatives that are not simultaneously 0. We require $d\mathbf{r}/dt \neq 0$ for a smooth curve to make sure the curve has a continuously turning tangent at each point. On a smooth curve, there are no sharp corners or cusps.

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth** (Figure 13.6).

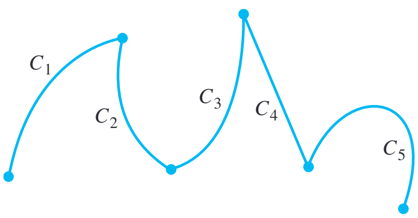



**FIGURE 13.6** A piecewise smooth curve made up of five smooth curves connected end to end in a continuous fashion. The curve here is not smooth at the points joining the five smooth curves.

Look once again at Figure 13.5. We drew the figure for $\Delta t$ positive, so $\Delta\mathbf{r}$ points forward, in the direction of the motion. The vector $\Delta\mathbf{r}/\Delta t$, having the same direction as $\Delta\mathbf{r}$, points forward too. Had $\Delta t$ been negative, $\Delta\mathbf{r}$ would have pointed backward, against the direction of motion. The quotient $\Delta\mathbf{r}/\Delta t$, however, being a negative scalar multiple of $\Delta\mathbf{r}$, would once again have pointed forward. No matter how $\Delta\mathbf{r}$ points, $\Delta\mathbf{r}/\Delta t$ points forward, and we expect the vector $d\mathbf{r}/dt = \lim_{\Delta t\to 0}\Delta\mathbf{r}/\Delta t$, when different from $\mathbf{0}$, to do the same. This means that the derivative $d\mathbf{r}/dt$, which is the rate of change of position with respect to time, always points in the direction of motion. For a smooth curve, $d\mathbf{r}/dt$ is never zero; the particle does not stop or reverse direction.

**DEFINITIONS** If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**. If $\mathbf{v}$ is a nonzero vector, then it is tangent to the curve, and its direction is the **direction of motion**. The magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time $t$.

**EXAMPLE 4** Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\cos^2 t\,\mathbf{k}$. Sketch the velocity vector $\mathbf{v}(7\pi/4)$.

**Solution** The velocity and acceleration vectors at time $t$ are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 10\cos t\sin t\,\mathbf{k}\\ &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k},\\ \mathbf{a}(t) &= \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k},\end{aligned}$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t}.$$

When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}, \qquad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

A sketch of the curve of motion, and the velocity vector when $t = 7\pi/4$, can be seen in Figure 13.7. ■

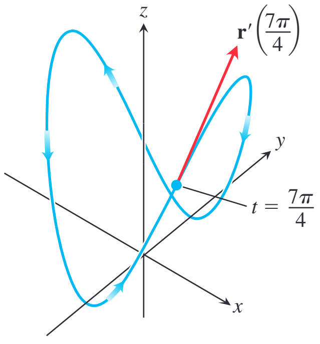



**FIGURE 13.7** The curve and the velocity vector when $t = 7\pi/4$ for the motion given in Example 4.

We can express the velocity of a moving particle as the product of its speed and direction:

$$\text{Velocity} = |\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = (\text{speed})(\text{direction}).$$

## Differentiation Rules

Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the same form as the rules for differentiating scalar functions.

### Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of $t$, **C** a constant vector, $c$ any scalar, and $f$ any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$
3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

When you use the Cross Product Rule, remember to preserve the order of the factors. If **u** comes first on the left side of the equation, it must also come first on the right, or the signs will be wrong.

We will prove the product rules and the Chain Rule but will leave the rules for constants, scalar multiples, sums, and differences as exercises.

**Proof of the Dot Product Rule** Suppose that

$$\mathbf{u} = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$$

and

$$\mathbf{v} = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.$$

Then

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3)\\ &= \underbrace{u_1'v_1 + u_2'v_2 + u_3'v_3}_{\mathbf{u}'\cdot\mathbf{v}} + \underbrace{u_1v_1' + u_2v_2' + u_3v_3'}_{\mathbf{u}\cdot\mathbf{v}'}.\end{aligned}$$

■

**Proof of the Cross Product Rule** We model the proof after the proof of the Product Rule for scalar functions. According to the definition of derivative,

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t)}{h}.$$

To change this fraction into an equivalent one that contains the difference quotients for the derivatives of **u** and **v**, we subtract and add $\mathbf{u}(t)\times\mathbf{v}(t+h)$ in the numerator. Then

$$\begin{aligned}&\frac{d}{dt}(\mathbf{u}\times\mathbf{v})\\ &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t+h) + \mathbf{u}(t)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t)}{h}\\ &= \lim_{h\to 0}\left[\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\mathbf{v}(t+h) + \mathbf{u}(t)\times\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}\right]\\ &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\lim_{h\to 0}\mathbf{v}(t+h) + \lim_{h\to 0}\mathbf{u}(t)\times\lim_{h\to 0}\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}.\end{aligned}$$

The last of these equalities holds because the limit of the cross product of two vector functions is the cross product of their limits if the latter exist (Exercise 46). As $h$ approaches zero, $\mathbf{v}(t+h)$ approaches $\mathbf{v}(t)$ because $\mathbf{v}$, being differentiable at $t$, is continuous at $t$ (Exercise 47). The two fractions approach the values of $d\mathbf{u}/dt$ and $d\mathbf{v}/dt$ at $t$. In short,

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \left(\frac{d\mathbf{u}}{dt}\times\mathbf{v}\right) + \left(\mathbf{u}\times\frac{d\mathbf{v}}{dt}\right). \qquad \blacksquare$$

**Proof of the Chain Rule** Suppose that $\mathbf{u}(s) = a(s)\mathbf{i} + b(s)\mathbf{j} + c(s)\mathbf{k}$ is a differentiable vector function of $s$ and that $s = f(t)$ is a differentiable scalar function of $t$. Then $a$, $b$, and $c$ are differentiable functions of $t$, and the Chain Rule for differentiable real-valued functions gives

$$\begin{aligned}
\frac{d}{dt}[\mathbf{u}(s)] &= \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} \\
&= \frac{da}{ds}\frac{ds}{dt}\mathbf{i} + \frac{db}{ds}\frac{ds}{dt}\mathbf{j} + \frac{dc}{ds}\frac{ds}{dt}\mathbf{k} \\
&= \frac{ds}{dt}\left(\frac{da}{ds}\mathbf{i} + \frac{db}{ds}\mathbf{j} + \frac{dc}{ds}\mathbf{k}\right) \\
&= \frac{ds}{dt}\frac{d\mathbf{u}}{ds} \\
&= f'(t)\,\mathbf{u}'(f(t)). \qquad s = f(t) \qquad \blacksquare
\end{aligned}$$

As an algebraic convenience, we sometimes write the product of a scalar $c$ and a vector $\mathbf{v}$ as $\mathbf{v}c$ instead of $c\mathbf{v}$. This permits us, for instance, to write the Chain Rule in a familiar form:

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}}{ds}\frac{ds}{dt},$$

where $s = f(t)$.

## Vector Functions of Constant Length

When we track a particle moving on a sphere centered at the origin (Figure 13.8), the position vector has a constant length equal to the radius of the sphere. The velocity vector $d\mathbf{r}/dt$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\mathbf{r}$. This is always the case for a differentiable vector function of constant length: The vector and its first derivative are orthogonal. By direct calculation,

$$\begin{aligned}
\mathbf{r}(t)\cdot\mathbf{r}(t) &= |\mathbf{r}(t)|^2 = c^2 && |\mathbf{r}(t)| = c \text{ is constant.} \\
\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] &= 0 && \text{Differentiate both sides.} \\
\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 && \text{Rule 5 with } \mathbf{r}(t) = \mathbf{u}(t) = \mathbf{v}(t) \\
2\mathbf{r}'(t)\cdot\mathbf{r}(t) &= 0.
\end{aligned}$$

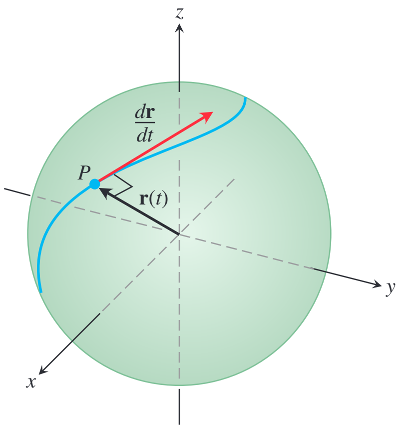



**FIGURE 13.8** If a particle moves on a sphere in such a way that its position $\mathbf{r}$ is a differentiable function of time, then $\mathbf{r}\cdot(d\mathbf{r}/dt) = 0$.

Thus the vectors $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal because their dot product is 0. In summary, the following holds.

> If $\mathbf{r}$ is a differentiable vector function of $t$ and the length of $\mathbf{r}(t)$ is constant, then
>
> $$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0. \qquad (4)$$

We will use this observation repeatedly in Section 13.4. The converse is also true (see Exercise 41).

## EXERCISES 13.1

In Exercises 1–4, find the given limits.

1. $\lim_{t\to\pi}\left[\left(\sin\frac{t}{2}\right)\mathbf{i}+\left(\cos\frac{2}{3}t\right)\mathbf{j}+\left(\tan\frac{5}{4}t\right)\mathbf{k}\right]$

2. $\lim_{t\to-1}\left[t^3\mathbf{i}+\left(\sin\frac{\pi}{2}t\right)\mathbf{j}+(\ln(t+2))\mathbf{k}\right]$

3. $\lim_{t\to1}\left[\left(\frac{t^2-1}{\ln t}\right)\mathbf{i}-\left(\frac{\sqrt{t}-1}{1-t}\right)\mathbf{j}+(\arctan t)\mathbf{k}\right]$

4. $\lim_{t\to0}\left[\left(\frac{\sin t}{t}\right)\mathbf{i}+\left(\frac{\tan^2 t}{\sin 2t}\right)\mathbf{j}-\left(\frac{t^3-8}{t+2}\right)\mathbf{k}\right]$

### Motion in the Plane

In Exercises 5–8, $\mathbf{r}(t)$ is the position of a particle in the $xy$-plane at time $t$. Find an equation in $x$ and $y$ whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of $t$.

5. $\mathbf{r}(t)=(t+1)\mathbf{i}+(t^2-1)\mathbf{j},\quad t=1$

6. $\mathbf{r}(t)=\frac{t}{t+1}\mathbf{i}+\frac{1}{t}\mathbf{j},\quad t=-\frac{1}{2}$

7. $\mathbf{r}(t)=e^t\mathbf{i}+\frac{2}{9}e^{2t}\mathbf{j},\quad t=\ln 3$

8. $\mathbf{r}(t)=(\cos 2t)\mathbf{i}+(3\sin 2t)\mathbf{j},\quad t=0$

Exercises 9–12 give the position vectors of particles moving along various curves in the $xy$-plane. In each case, find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

9. **Motion on the circle $x^2+y^2=1$**

$$\mathbf{r}(t)=(\sin t)\mathbf{i}+(\cos t)\mathbf{j};\quad t=\pi/4 \text{ and } \pi/2$$

10. **Motion on the circle $x^2+y^2=16$**

$$\mathbf{r}(t)=\left(4\cos\frac{t}{2}\right)\mathbf{i}+\left(4\sin\frac{t}{2}\right)\mathbf{j};\quad t=\pi \text{ and } 3\pi/2$$

11. **Motion on the cycloid $x=t-\sin t,\quad y=1-\cos t$**

$$\mathbf{r}(t)=(t-\sin t)\mathbf{i}+(1-\cos t)\mathbf{j};\quad t=\pi \text{ and } 3\pi/2$$

12. **Motion on the parabola $y=x^2+1$**

$$\mathbf{r}(t)=t\mathbf{i}+(t^2+1)\mathbf{j};\quad t=-1, 0, \text{ and } 1$$

### Motion in Space

In Exercises 13–18, $\mathbf{r}(t)$ is the position of a particle in space at time $t$. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of $t$. Write the particle's velocity at that time as the product of its speed and direction.

13. $\mathbf{r}(t)=(t+1)\mathbf{i}+(t^2-1)\mathbf{j}+2t\mathbf{k},\quad t=1$

14. $\mathbf{r}(t)=(1+t)\mathbf{i}+\frac{t^2}{\sqrt{2}}\mathbf{j}+\frac{t^3}{3}\mathbf{k},\quad t=1$

15. $\mathbf{r}(t)=(2\cos t)\mathbf{i}+(3\sin t)\mathbf{j}+4t\mathbf{k},\quad t=\pi/2$

16. $\mathbf{r}(t)=(\sec t)\mathbf{i}+(\tan t)\mathbf{j}+\frac{4}{3}t\mathbf{k},\quad t=\pi/6$

17. $\mathbf{r}(t)=(2\ln(t+1))\mathbf{i}+t^2\mathbf{j}+\frac{t^2}{2}\mathbf{k},\quad t=1$

18. $\mathbf{r}(t)=e^{-t}\mathbf{i}+(2\cos 3t)\mathbf{j}+(2\sin 3t)\mathbf{k},\quad t=0$

In Exercises 19–22, $\mathbf{r}(t)$ is the position of a particle in space at time $t$. Find the angle between the velocity and acceleration vectors at time $t=0$.

19. $\mathbf{r}(t)=(3t+1)\mathbf{i}+\sqrt{3}t\mathbf{j}+t^2\mathbf{k}$

20. $\mathbf{r}(t)=\left(\frac{\sqrt{2}}{2}t\right)\mathbf{i}+\left(\frac{\sqrt{2}}{2}t-16t^2\right)\mathbf{j}$

21. $\mathbf{r}(t)=(\ln(t^2+1))\mathbf{i}+(\arctan t)\mathbf{j}+\sqrt{t^2+1}\,\mathbf{k}$

22. $\mathbf{r}(t)=\frac{4}{9}(1+t)^{3/2}\mathbf{i}+\frac{4}{9}(1-t)^{3/2}\mathbf{j}+\frac{1}{3}t\mathbf{k}$

### Tangents to Curves

As mentioned in the text, the **tangent line** to a smooth curve $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$ at $t=t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at $t_0$. In Exercises 23–26, find parametric equations for the line that is tangent to the given curve at the given parameter value $t=t_0$.

23. $\mathbf{r}(t)=(\sin t)\mathbf{i}+(t^2-\cos t)\mathbf{j}+e^t\mathbf{k},\quad t_0=0$

24. $\mathbf{r}(t)=t^2\mathbf{i}+(2t-1)\mathbf{j}+t^3\mathbf{k},\quad t_0=2$

25. $\mathbf{r}(t)=\ln t\,\mathbf{i}+\frac{t-1}{t+2}\mathbf{j}+t\ln t\,\mathbf{k},\quad t_0=1$

26. $\mathbf{r}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+(\sin 2t)\mathbf{k},\quad t_0=\frac{\pi}{2}$

In Exercises 27–30, find the value(s) of $t$ so that the tangent line to the given curve contains the given point.

27. $\mathbf{r}(t)=t^2\mathbf{i}+(1+t)\mathbf{j}+(2t-3)\mathbf{k};\quad (-8, 2, -1)$

28. $\mathbf{r}(t)=t\mathbf{i}+3\mathbf{j}+\left(\frac{2}{3}t^{3/2}\right)\mathbf{k};\quad (0, 3, -8/3)$

29. $\mathbf{r}(t)=2t\mathbf{i}+t^2\mathbf{j}-t^2\mathbf{k};\quad (0, -4, 4)$

30. $\mathbf{r}(t)=-t\mathbf{i}+t^2\mathbf{j}+(\ln t)\mathbf{k};\quad (2, -5, -3)$

In Exercises 31–36, $\mathbf{r}(t)$ is the position of a particle in space at time $t$. Match each position function with one of the graphs A–F.

31. $\mathbf{r}(t)=(t\cos t)\mathbf{i}+(t\sin t)\mathbf{j}+t\mathbf{k}$

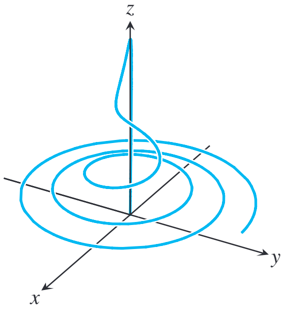
32. $\mathbf{r}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+(\sin 2t)\mathbf{k}$

33. $\mathbf{r}(t)=t^2\mathbf{i}+(t^2+1)\mathbf{j}+t^4\mathbf{k}$

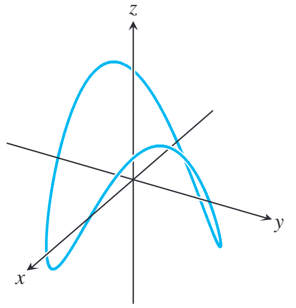
34. $\mathbf{r}(t)=t\mathbf{i}+(\ln t)\mathbf{j}+(\sin t)\mathbf{k}$

**35.** $\mathbf{r}(t) = t\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}$

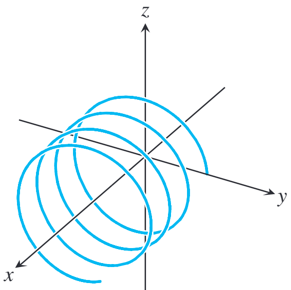
**36.** $\mathbf{r}(t) = (t \sin t)\mathbf{i} + (t \cos t)\mathbf{j} + \left(\dfrac{t}{t^2 + 1}\right)\mathbf{k}$



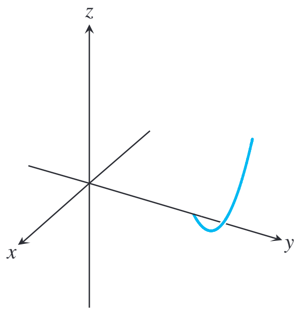

A.



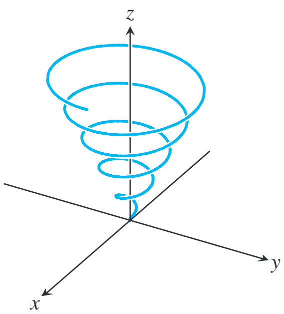

B.



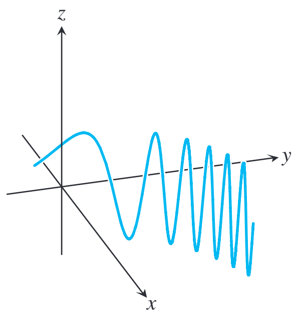

C.




D.




E.




F.

## Theory and Examples

**37. Motion along a circle** Each of the following equations in parts (a)–(e) describes the motion of a particle having the same path, namely the unit circle $x^2 + y^2 = 1$. Although the path of each particle in parts (a)–(e) is the same, the behavior, or "dynamics," of each particle is different. For each particle, answer the following questions.

**i)** Does the particle have constant speed? If so, what is its constant speed?

**ii)** Is the particle's acceleration vector always orthogonal to its velocity vector?

**iii)** Does the particle move clockwise or counterclockwise around the circle?

**iv)** Is the particle initially located at the point $(1, 0)$?

**a.** $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad t \geq 0$

**b.** $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}, \quad t \geq 0$

**c.** $\mathbf{r}(t) = \cos(t - \pi/2)\mathbf{i} + \sin(t - \pi/2)\mathbf{j}, \quad t \geq 0$

**d.** $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad t \geq 0$

**e.** $\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j}, \quad t \geq 0$

**38. Motion along a circle** Show that the vector-valued function

$$\mathbf{r}(t) = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \cos t\left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) + \sin t\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$$

describes the motion of a particle moving in the circle of radius 1 centered at the point $(2, 2, 1)$ and lying in the plane $x + y - 2z = 2$.

**39. Motion along a parabola** A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point $(2, 2)$.

**40. Motion along a cycloid** A particle moves in the $xy$-plane in such a way that its position at time $t$ is

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}.$$

T **a.** Graph $\mathbf{r}(t)$. The resulting curve is a cycloid.

**b.** Find the maximum and minimum values of $|\mathbf{v}|$ and $|\mathbf{a}|$. (*Hint:* Find the extreme values of $|\mathbf{v}|^2$ and $|\mathbf{a}|^2$ first, and take square roots later.)

**41.** Let $\mathbf{r}$ be a differentiable vector function of $t$. Show that if $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$ for all $t$, then $|\mathbf{r}|$ is constant.

**42. Derivatives of triple scalar products**

**a.** Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable vector functions of $t$, then

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} \times \mathbf{w} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}.$$

**b.** Show that

$$\frac{d}{dt}\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3}\right).$$

(*Hint:* Differentiate on the left and look for vectors whose products are zero.)

**43.** Prove the two Scalar Multiple Rules for vector functions.

**44.** Prove the Sum and Difference Rules for vector functions.

**45. Component test for continuity at a point** Show that the vector function $\mathbf{r}$ defined by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at $t = t_0$ if and only if $f$, $g$, and $h$ are continuous at $t_0$.

**46. Limits of cross products of vector functions** Suppose that $\mathbf{r}_1(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, $\mathbf{r}_2(t) = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$, $\lim_{t \to t_0} \mathbf{r}_1(t) = \mathbf{A}$, and $\lim_{t \to t_0} \mathbf{r}_2(t) = \mathbf{B}$. Use the determinant formula for cross products and the Limit Product Rule for scalar functions to show that

$$\lim_{t \to t_0} (\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{A} \times \mathbf{B}.$$

**47. Differentiable vector functions are continuous** Show that if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is differentiable at $t = t_0$, then it is continuous at $t_0$ as well.

**48. Constant Function Rule** Prove that if $\mathbf{u}$ is the vector function with the constant value $\mathbf{C}$, then $d\mathbf{u}/dt = \mathbf{0}$.

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 49–52.

**a.** Plot the space curve traced out by the position vector $\mathbf{r}$.

**b.** Find the components of the velocity vector $d\mathbf{r}/dt$.

**c.** Evaluate $d\mathbf{r}/dt$ at the given point $t_0$ and determine the equation of the tangent line to the curve at $\mathbf{r}(t_0)$.

**d.** Plot the tangent line together with the curve over the given interval.

**49.** $\mathbf{r}(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 6\pi$, $t_0 = 3\pi/2$

**50.** $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $-2 \le t \le 3$, $t_0 = 1$

**51.** $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\ln(1 + t))\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4\pi$, $t_0 = \pi/4$

**52.** $\mathbf{r}(t) = (\ln(t^2 + 2))\mathbf{i} + (\arctan 3t)\mathbf{j} + \sqrt{t^2 + 1}\,\mathbf{k}$, $-3 \le t \le 5$, $t_0 = 3$

In Exercises 53 and 54, you will explore graphically the behavior of the helix

$$\mathbf{r}(t) = (\cos at)\mathbf{i} + (\sin at)\mathbf{j} + bt\mathbf{k}$$

as you change the values of the constants $a$ and $b$. Use a CAS to perform the steps in each exercise.

**53.** Set $b = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $a = 1, 2, 4$, and 6 over the interval $0 \le t \le 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as $a$ increases through these positive values.

**54.** Set $a = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $b = 1/4, 1/2, 2$, and 4 over the interval $0 \le t \le 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as $b$ increases through these positive values.

# 13.2 Integrals of Vector Functions; Projectile Motion

In this section we investigate integrals of vector functions and their application to motion along a path in space or in the plane.

## Integrals of Vector Functions

A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval $I$ if $d\mathbf{R}/dt = \mathbf{r}$ at each point of $I$. If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on $I$, it can be shown, working one component at a time, that every antiderivative of $\mathbf{r}$ on $I$ has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$ (Exercise 45). The set of all antiderivatives of $\mathbf{r}$ on $I$ is the **indefinite integral** of $\mathbf{r}$ on $I$.

**DEFINITION** The **indefinite integral** of $\mathbf{r}$ with respect to $t$ is the set of all antiderivatives of $\mathbf{r}$, denoted by

$$\int \mathbf{r}(t)\,dt.$$

The usual arithmetic rules for indefinite integrals apply.

**EXAMPLE 1** To integrate a vector function, we integrate each of its components.

$$\int((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int \cos t\,dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\,dt\right)\mathbf{k} \quad (1)$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \quad (2)$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \qquad C = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$