**27.** Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

**a.** $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

**b.** $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$

**c.** $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

**d.** $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$

**e.** $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$

**f.** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

**g.** $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

**h.** $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

**28.** Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

**a.** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

**b.** $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

**c.** $(-\mathbf{u}) \times \mathbf{v} = -(\mathbf{u} \times \mathbf{v})$

**d.** $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (any number $c$)

**e.** $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (any number $c$)

**f.** $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

**g.** $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = 0$

**h.** $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

**29.** Given nonzero vectors **u**, **v**, and **w**, use dot product and cross product notation, as appropriate, to describe the following.

**a.** The vector projection of **u** onto **v**

**b.** A vector orthogonal to **u** and **v**

**c.** A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and **w**

**d.** The volume of the parallelepiped determined by **u**, **v**, and **w**

**e.** A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$

**f.** A vector of length $|\mathbf{u}|$ in the direction of **v**

**30.** Compute $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{j} \times \mathbf{j})$. What can you conclude about the associativity of the cross product?

**31.** Let **u**, **v**, and **w** be vectors. Which of the following make sense, and which do not? Give reasons for your answers.

**a.** $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ **b.** $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

**c.** $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ **d.** $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

**32. Cross products of three vectors** Show that except in degenerate cases, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of **u** and **v**, whereas $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of **v** and **w**. What *are* the degenerate cases?

**33. Cancelation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

**34. Double cancelation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

### Area of a Parallelogram

Find the areas of the parallelograms whose vertices are given in Exercises 35–40.

**35.** $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$

**36.** $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$

**37.** $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$

**38.** $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$

**39.** $A(0, 0, 0)$, $B(3, 2, 4)$, $C(5, 1, 4)$, $D(2, -1, 0)$

**40.** $A(1, 0, -1)$, $B(1, 7, 2)$, $C(2, 4, -1)$, $D(0, 3, 2)$

### Area of a Triangle

Find the areas of the triangles whose vertices are given in Exercises 41–47.

**41.** $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$

**42.** $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$

**43.** $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$

**44.** $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$

**45.** $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -1)$

**46.** $A(0, 0, 0)$, $B(-1, 1, -1)$, $C(3, 0, 3)$

**47.** $A(1, -1, 1)$, $B(0, 1, 1)$, $C(1, 0, -1)$

**48.** Find the volume of a parallelepiped with one of its eight vertices at $A(0, 0, 0)$ and three adjacent vertices at $B(1, 2, 0)$, $C(0, -3, 2)$, and $D(3, -4, 5)$.

**49. Triangle area** Find a $2 \times 2$ determinant formula for the area of the triangle in the $xy$-plane with vertices at $(0, 0)$, $(a_1, a_2)$, and $(b_1, b_2)$. Explain your work.

**50. Triangle area** Find a concise $3 \times 3$ determinant formula that gives the area of a triangle in the $xy$-plane having vertices $(a_1, a_2)$, $(b_1, b_2)$, and $(c_1, c_2)$.

### Volume of a Tetrahedron

Using the methods of Section 6.1, where volume is computed by integrating cross-sectional area, it can be shown that the volume of a tetrahedron formed by three vectors is equal to $\frac{1}{6}$ the volume of the parallelepiped formed by the three vectors. Find the volumes of the tetrahedra whose vertices are given in Exercises 51–54.

**51.** $A(0, 0, 0)$, $B(2, 0, 0)$, $C(0, 3, 0)$, $D(0, 0, 4)$

**52.** $A(0, 0, 0)$, $B(1, 0, 2)$, $C(0, 2, 1)$, $D(3, 4, 0)$

**53.** $A(1, -1, 0)$, $B(0, 2, -2)$, $C(-3, 0, 3)$, $D(0, 4, 4)$

**54.** $A(-1, 2, 3)$, $B(2, 0, 1)$, $C(1, -3, 2)$, $D(-2, 1, -1)$

In Exercises 55–57, determine whether the given points are coplanar.

**55.** $A(1, 1, 1)$, $B(-1, 0, 4)$, $C(0, 2, 1)$, $D(2, -2, 3)$

**56.** $A(0, 0, 4)$, $B(6, 2, 0)$, $C(2, -1, 1)$, $D(-3, -4, 3)$

**57.** $A(0, 1, 2)$, $B(-1, 1, 0)$, $C(2, 0, -1)$, $D(1, -1, 1)$

## 12.5 Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space. We will use these representations throughout the rest of the text in studying the calculus of curves and surfaces in space.

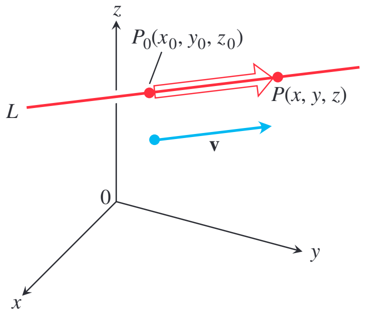



**FIGURE 12.37** A point $P$ lies on $L$ through $P_0$ parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.

## Lines and Line Segments in Space

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a *vector* giving the direction of the line.

Suppose that $L$ is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then $L$ is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$ (Figure 12.37). Thus, $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar parameter $t$. The value of $t$ depends on the location of the point $P$ along the line, and the domain of $t$ is $(-\infty, \infty)$. The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \tag{1}$$

If $\mathbf{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and $\mathbf{r}_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

### Vector Equation for a Line

**A vector equation for the line $L$ through $P_0(x_0, y_0, z_0)$ parallel to a nonzero vector $\mathbf{v}$** is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on $L$ and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Equating the corresponding components of the two sides of Equation (1) gives three scalar equations involving the parameter $t$:

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

These equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$.

### Parametric Equations for a Line

**The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to a nonzero vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$** is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty. \tag{3}$$

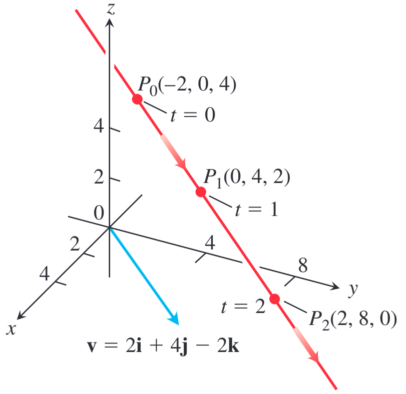



**FIGURE 12.38** Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing $t$.

**EXAMPLE 1** Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure 12.38).

**Solution** With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t. \qquad \blacksquare$$

**EXAMPLE 2** Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

**Solution** The vector

$$\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of $t$. ■

Notice that parametrizations are not unique. Not only can the "base point" change, but so can the parameter. The equations $x = -3 + 4t^3$, $y = 2 - 3t^3$, and $z = -3 + 7t^3$ also parametrize the line in Example 2.

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the $t$-values for the endpoints and restrict $t$ to lie in the closed interval bounded by these values. The line equations, together with this added restriction, parametrize the segment.

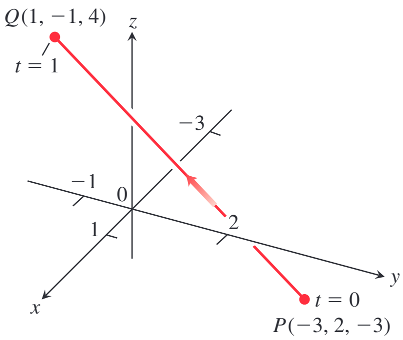



**FIGURE 12.39** Example 3 derives a parametrization of line segment $PQ$. The arrow shows the direction of increasing $t$.

**EXAMPLE 3** Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.39).

**Solution** We begin with equations for the line through $P$ and $Q$, taking them, in this case, from Example 2:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

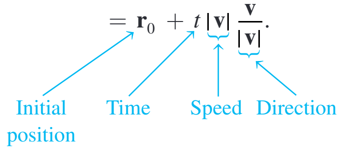
$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \le t \le 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \le t \le 1. \quad ■$$

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector $\mathbf{v}$. Rewriting Equation (2), we have

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v} \\ &= \mathbf{r}_0 + t\,|\mathbf{v}|\,\frac{\mathbf{v}}{|\mathbf{v}|}. \end{aligned} \tag{4}$$




In other words, the position of the particle at time $t$ is its initial position plus its distance moved (speed × time) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight-line motion.

**EXAMPLE 4** A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

**Solution** We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time $t$ is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\,\mathbf{u}\\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)\\ &= 20\sqrt{3}\,t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})\\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\rangle.\end{aligned}$$

After 10 sec of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of $(60\text{ ft/sec})(10\text{ sec}) = 600$ ft, which is the length of the vector $\mathbf{r}(10)$. ■

## The Distance from a Point to a Line in Space

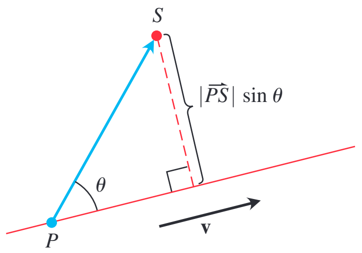



**FIGURE 12.40** The distance from $S$ to the line through $P$ parallel to $\mathbf{v}$ is $|\overrightarrow{PS}|\sin\theta$, where $\theta$ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

To find the distance from a point $S$ to a line that passes through a point $P$ parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line (Figure 12.40). In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}|\sin\theta$, which is $\dfrac{|\overrightarrow{PS}|\,|\mathbf{v}|\sin\theta}{|\mathbf{v}|} = \dfrac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$.

**Distance from a Point $S$ to a Line Through $P$ Parallel to v**

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad (5)$$

**EXAMPLE 5** Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

**Solution** We see from the equations for $L$ that $L$ passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1-1)\mathbf{i} + (1-3)\mathbf{j} + (5-0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -2 & 5\\ 1 & -1 & 2\end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$ ■

## An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular, or normal, to the plane.

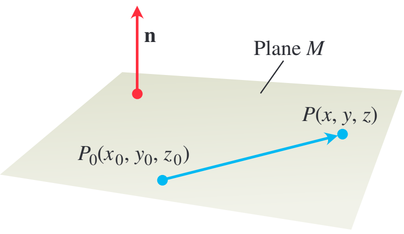



**FIGURE 12.41** The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point $P$ lies in the plane through $P_0$ normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Suppose that plane $M$ passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. A vector from $P_0$ to any point $P$ on the plane is orthogonal to $\mathbf{n}$. Then $M$ is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (Figure 12.41). Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0,$$

so the plane $M$ consists of the points $(x, y, z)$ satisfying

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

### Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to a nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

**Vector equation:** $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

**Component equation:** $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

**Component equation simplified:** $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

**EXAMPLE 6** Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

**Solution** The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$\begin{aligned} 5x + 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of $x$, $y$, and $z$ in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

**EXAMPLE 7** Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

**Solution** We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$\begin{aligned} 3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6. \end{aligned}$$

## Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar $k$. Two planes that are not parallel intersect in a line.

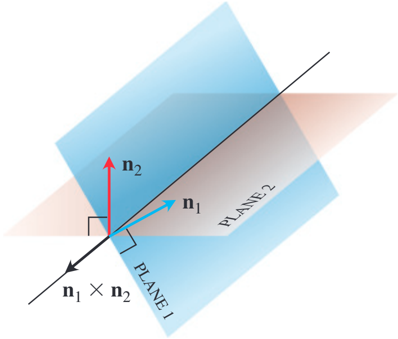



**FIGURE 12.42** How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

**EXAMPLE 8** Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

**Solution** The line of intersection of two planes is perpendicular to both planes' normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (Figure 12.42) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ■

**EXAMPLE 9** Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

**Solution** We find a vector parallel to the line and a point on the line and use Equations (3).

Example 8 identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for $x$ and $y$ simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

The choice $z = 0$ is arbitrary, and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for $y$ and $z$. The different choices would simply give different parametrizations of the same line. ■

Sometimes we want to know where a line and a plane intersect. For example, if we are looking at a flat plate and a line segment passes through it, we may be interested in knowing what portion of the line segment is hidden from our view by the plate. This application is used in computer graphics (Exercise 78).

**EXAMPLE 10** Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

**Solution** The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane—that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$ ■

## The Distance from a Point to a Plane

If $P$ is a point on a plane with a normal $\mathbf{n}$, then the distance from any point $S$ to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$, as given in the following formula.

**Distance from a Point $S$ to a Plane Through a Point $P$ with a Normal $\mathbf{n}$**

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \tag{6}$$

**EXAMPLE 11** Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

**Solution** We find a point $P$ in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\mathbf{n}$ normal to the plane (Figure 12.43). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

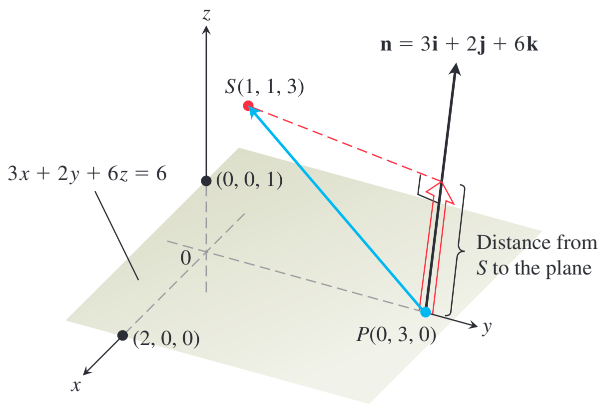



**FIGURE 12.43** The distance from $S$ to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$ (Example 11).

The points on the plane easiest to find from the plane's equation are the intercepts. If we take $P$ to be the $y$-intercept $(0, 3, 0)$, then

$$\overrightarrow{PS} = (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$

$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

Therefore, the distance from $S$ to the plane is

$$\begin{aligned} d &= \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| && \text{Length of } \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} \\ &= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left( \frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}. \end{aligned}$$

■

## Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.44).

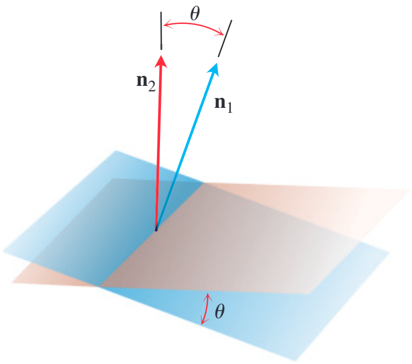



**FIGURE 12.44** The angle between two planes is obtained from the angle between their normals.

**EXAMPLE 12** Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

**Solution** The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)\\ &= \arccos\left(\frac{4}{21}\right) \approx 1.38 \text{ radians.} \quad \text{About 79 degrees}\end{aligned}$$

## EXERCISES 12.5

### Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t$, $y = 2 - t$, $z = 3t$
7. The line through $(1, 1, 1)$ parallel to the $z$-axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$
10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$
11. The $x$-axis
12. The $z$-axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing $t$ for your parametrization.

13. $(0, 0, 0)$, $(1, 1, 3/2)$
14. $(0, 0, 0)$, $(1, 0, 0)$
15. $(1, 0, 0)$, $(1, 1, 0)$
16. $(1, 1, 0)$, $(1, 1, 1)$
17. $(0, 1, 1)$, $(0, -1, 1)$
18. $(0, 2, 0)$, $(3, 0, 0)$
19. $(2, 0, 2)$, $(0, 2, 0)$
20. $(1, 0, -1)$, $(0, 3, 0)$

### Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
22. The plane through $(1, -1, 3)$ parallel to the plane
$$3x + y + z = 7$$
23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$
25. The plane through $P_0(2, 4, 5)$ perpendicular to the line
$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$
26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to $A$.
27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.
28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane containing the intersecting lines.

29. $L1$: $x = -1 + t$, $y = 2 + t$, $z = 1 - t$; $-\infty < t < \infty$
$L2$: $x = 1 - 4s$, $y = 1 + 2s$, $z = 2 - 2s$; $-\infty < s < \infty$
30. $L1$: $x = t$, $y = 3 - 3t$, $z = -2 - t$; $-\infty < t < \infty$
$L2$: $x = 1 + s$, $y = 4 + s$, $z = -1 + s$; $-\infty < s < \infty$
31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.
32. Find a plane through the points $P_1(1, 2, 3)$, and $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

### Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12)$; $x = 4t$, $y = -2t$, $z = 2t$
34. $(0, 0, 0)$; $x = 5 + 3t$, $y = 5 + 4t$, $z = -3 - 5t$
35. $(2, 1, 3)$; $x = 2 + 2t$, $y = 1 + 6t$, $z = 3$
36. $(2, 1, -1)$; $x = 2t$, $y = 1 + 2t$, $z = 2t$
37. $(3, -1, 4)$; $x = 4 - t$, $y = 3 + 2t$, $z = -5 + 3t$
38. $(-1, 4, 3)$; $x = 10 + 4t$, $y = -3$, $z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $x + 2y + 2z = 13$
40. $(0, 0, 0)$, $3x + 2y + 6z = 6$
41. $(0, 1, 1)$, $4y + 3z = -12$

**42.** $(2, 2, 3), \quad 2x + y + 2z = 4$

**43.** $(0, -1, 0), \quad 2x + y + 2z = 4$

**44.** $(1, 0, -1), \quad -4x + y + z = 4$

**45.** Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

**46.** Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

### Angles

In Exercises 47 and 48, find the angles between the planes.

**47.** $x + y = 1, \quad 2x + y - 2z = 2$

**48.** $5x + y - z = 10, \quad x - 2y + 3z = -1$

In Exercises 49 and 50, find the acute angles between the intersecting lines.

**49.** $x = t, y = 2t, z = -t$ and $x = 1 - t, y = 5 + t, z = 2t$

**50.** $x = 2 + t, y = 4t + 2, z = 1 + t$ and $x = 3t - 2, y = -2, z = 2 - 2t$

In Exercises 51 and 52, find the acute angles between the lines and planes.

**51.** $x = 1 - t, y = 3t, z = 1 + t; \quad 2x - y + 3z = 6$

**52.** $x = 2, y = 3 + 2t, z = 1 - 2t; \quad x - y + z = 0$

T Use a calculator to find the acute angles between the planes in Exercises 53–56 to the nearest hundredth of a radian.

**53.** $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$

**54.** $x + y + z = 1, \quad z = 0$ (the $xy$-plane)

**55.** $2x + 2y - z = 3, \quad x + 2y + z = 2$

**56.** $4y + 3z = -12, \quad 3x + 2y + 6z = 6$

### Intersecting Lines and Planes

In Exercises 57–60, find the point in which the line meets the plane.

**57.** $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

**58.** $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

**59.** $x = 1 + 2t, \quad y = 1 + 5t, \quad z = 3t; \quad x + y + z = 2$

**60.** $x = -1 + 3t, \quad y = -2, \quad z = 5t; \quad 2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 61–64 intersect.

**61.** $x + y + z = 1, \quad x + y = 2$

**62.** $3x - 6y - 2z = 3, \quad 2x + y - 2z = 2$

**63.** $x - 2y + 4z = 2, \quad x + y - 2z = 5$

**64.** $5x - 2y = 11, \quad 4y - 5z = -17$

Given two lines in space, either they are parallel, they intersect, or they are skew (lie in parallel planes). In Exercises 65 and 66, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection. Otherwise, find the distance between the two lines.

**65.** $L1$: $x = 3 + 2t, y = -1 + 4t, z = 2 - t; \quad -\infty < t < \infty$

$L2$: $x = 1 + 4s, y = 1 + 2s, z = -3 + 4s; \quad -\infty < s < \infty$

$L3$: $x = 3 + 2r, y = 2 + r, z = -2 + 2r; \quad -\infty < r < \infty$

**66.** $L1$: $x = 1 + 2t, \quad y = -1 - t, \quad z = 3t; \quad -\infty < t < \infty$

$L2$: $x = 2 - s, \quad y = 3s, \quad z = 1 + s; \quad -\infty < s < \infty$

$L3$: $x = 5 + 2r, \quad y = 1 - r, \quad z = 8 + 3r; \quad -\infty < r < \infty$

### Theory and Examples

**67.** Use Equations (3) to generate a parametrization of the line through $P(2, -4, 7)$ parallel to $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then generate another parametrization of the line using the point $P_2(-2, -2, 1)$ and the vector $\mathbf{v}_2 = -\mathbf{i} + (1/2)\mathbf{j} - (3/2)\mathbf{k}$.

**68.** Use the component form to generate an equation for the plane through $P_1(4, 1, 5)$ normal to $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then generate another equation for the same plane using the point $P_2(3, -2, 0)$ and the normal vector $\mathbf{n}_2 = -\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$.

**69.** Find the points in which the line $x = 1 + 2t, \ y = -1 - t, \ z = 3t$ meets the coordinate planes. Describe the reasoning behind your answer.

**70.** Find equations for the line in the plane $z = 3$ that makes an angle of $\pi/6$ rad with $\mathbf{i}$ and an angle of $\pi/3$ rad with $\mathbf{j}$. Describe the reasoning behind your answer.

**71.** Is the line $x = 1 - 2t, \ y = 2 + 5t, \ z = -3t$ parallel to the plane $2x + y - z = 8$? Give reasons for your answer.

**72.** How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Give reasons for your answer.

**73.** Find two different planes whose intersection is the line $x = 1 + t, \ y = 2 - t, \ z = 3 + 2t$. Write equations for each plane in the form $Ax + By + Cz = D$.

**74.** Find a plane through the origin that is perpendicular to the plane $M: 2x + 3y + z = 12$ in a right angle. How do you know that your plane is perpendicular to $M$?

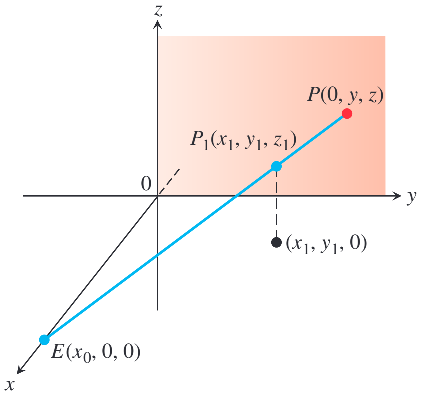
**75.** The graph of $(x/a) + (y/b) + (z/c) = 1$ is a plane for any nonzero numbers $a$, $b$, and $c$. Which planes have an equation of this form?

**76.** Suppose $L_1$ and $L_2$ are disjoint (nonintersecting) nonparallel lines. Is it possible for a nonzero vector to be perpendicular to both $L_1$ and $L_2$? Give reasons for your answer.

**77. Perspective in computer graphics** In computer graphics and perspective drawing, we need to represent objects seen by the eye in space as images on a two-dimensional plane. Suppose that the eye is at $E(x_0, 0, 0)$ as shown here and that we want to represent a point $P_1(x_1, y_1, z_1)$ as a point on the $yz$-plane. We do this by projecting $P_1$ onto the plane with a ray from $E$. The point $P_1$ will be portrayed as the point $P(0, y, z)$. The problem for us as graphics designers is to find $y$ and $z$ given $E$ and $P_1$.

**a.** Write a vector equation that holds between $\overrightarrow{EP}$ and $\overrightarrow{EP_1}$. Use the equation to express $y$ and $z$ in terms of $x_0$, $x_1$, $y_1$, and $z_1$.

**b.** Test the formulas obtained for $y$ and $z$ in part (a) by investigating their behavior at $x_1 = 0$ and $x_1 = x_0$ and by seeing what happens as $x_0 \to \infty$. What do you find?



**78. Hidden lines in computer graphics** Here is another typical problem in computer graphics. Your eye is at $(4, 0, 0)$. You are looking at a triangular plate whose vertices are at $(1, 0, 1)$, $(1, 1, 0)$, and $(-2, 2, 2)$. The line segment from $(1, 0, 0)$ to $(0, 2, 2)$ passes through the plate. What portion of the line segment is hidden from your view by the plate? (This is an exercise in finding intersections of lines and planes.)

## 12.6 Cylinders and Quadric Surfaces

Up to now, we have studied two special types of surfaces: spheres and planes. In this section, we extend our inventory to include a variety of cylinders and quadric surfaces. Quadric surfaces are surfaces defined by second-degree equations in $x$, $y$, and $z$. Spheres are quadric surfaces, but there are others of equal interest that will be needed in Chapters 14–16.

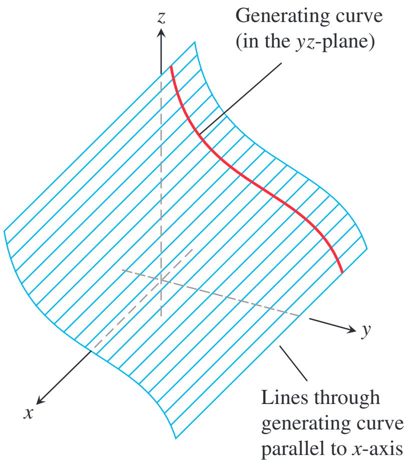



**FIGURE 12.45** A cylinder and generating curve.

### Cylinders

Suppose we are given a plane in space that contains a curve, and in addition we are given a line that is not parallel to this plane. A **cylinder** is a surface that is generated by moving a line that is parallel to the given line along the curve, while keeping it parallel to the given line. The curve is called a **generating curve** for the cylinder (Figure 12.45 illustrates this when the given plane is the $yz$-plane and the given line is the $x$-axis). In solid geometry, where *cylinder* means *circular cylinder*, the generating curves are circles, but now we allow generating curves of any kind. The cylinder in our first example is generated by a parabola.

**EXAMPLE 1** Find an equation for the cylinder made by the lines parallel to the $z$-axis that pass through the parabola $y = x^2$, $z = 0$ (Figure 12.46).

**Solution** The point $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the $xy$-plane. Then, for any value of $z$, the point $Q(x_0, x_0^2, z)$ lies on the cylinder because it lies on the line $x = x_0$, $y = x_0^2$ through $P_0$ parallel to the $z$-axis. Conversely, any point $Q(x_0, x_0^2, z)$ whose $y$-coordinate is the square of its $x$-coordinate lies on the cylinder because it lies on the line $x = x_0$, $y = x_0^2$ through $P_0$ parallel to the $z$-axis (Figure 12.46).

Regardless of the value of $z$, therefore, the points on the surface are the points whose coordinates satisfy the equation $y = x^2$. This makes $y = x^2$ an equation for the cylinder. ■

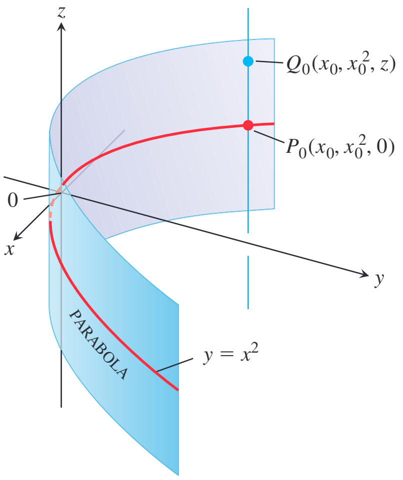



**FIGURE 12.46** Every point of the cylinder in Example 1 has coordinates of the form $(x_0, x_0^2, z)$.

As Example 1 suggests, any curve $f(x, y) = c$ in the $xy$-plane generates a cylinder parallel to the $z$-axis whose equation is also $f(x, y) = c$. For instance, the equation $x^2 + y^2 = 1$ corresponds to the circular cylinder made by the lines parallel to the $z$-axis that pass through the circle $x^2 + y^2 = 1$ in the $xy$-plane.

In a similar way, any curve $g(x, z) = c$ in the $xz$-plane generates a cylinder parallel to the $y$-axis whose space equation is also $g(x, z) = c$. Any curve $h(y, z) = c$ generates a cylinder parallel to the $x$-axis whose space equation is also $h(y, z) = c$. The axis of a cylinder need not be parallel to a coordinate axis, however.

### Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equation in $x$, $y$, and $z$. We first focus on quadric surfaces given by the equation

$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

where $A$, $B$, $C$, $D$, and $E$ are constants. The basic quadric surfaces are **ellipsoids**, **paraboloids**, **elliptical cones**, and **hyperboloids**. Spheres are special cases of ellipsoids. We present a few examples illustrating how to sketch a quadric surface, and then we give a summary table of graphs of the basic types.