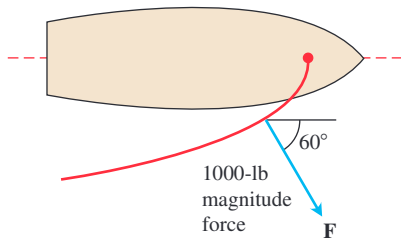


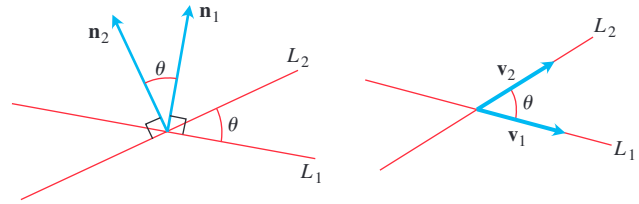
Work

- 43. Work along a line** Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters).
- 44. Locomotive** The Union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?
- 45. Inclined plane** How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200-N force at an angle of 30° from the horizontal?
- 46. Sailboat** The wind passing over a boat's sail exerted a 1000-lb magnitude force \mathbf{F} as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Angles Between Lines in the Plane

The acute angle between intersecting lines that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by vectors parallel to the lines.



Use this fact and the results of Exercise 33 or 34 to find the acute angles between the lines in Exercises 47–52.

- 47.** $3x + y = 5, \quad 2x - y = 4$
48. $y = \sqrt{3}x - 1, \quad y = -\sqrt{3}x + 2$
49. $\sqrt{3}x - y = -2, \quad x - \sqrt{3}y = 1$
50. $x + \sqrt{3}y = 1, \quad (1 - \sqrt{3})x + (1 + \sqrt{3})y = 8$
51. $3x - 4y = 3, \quad x - y = 7$
52. $12x + 5y = 1, \quad 2x - 2y = 3$

Dot Products of n -Dimensional Vectors

In Exercises 53–56, (a) find $\mathbf{u} \cdot \mathbf{v}$ and (b) determine whether the vectors \mathbf{u} and \mathbf{v} are orthogonal.

- 53.** $\mathbf{u} = \langle 3, 2, -4, 0 \rangle, \quad \mathbf{v} = \langle 1, 0, 0, 2 \rangle$
54. $\mathbf{u} = \langle -2, 1, 1, 2 \rangle, \quad \mathbf{v} = \langle -1, 2, -2, -1 \rangle$
55. $\mathbf{u} = \langle 6, 3, 0, 1, -2 \rangle, \quad \mathbf{v} = \langle 0, 2, -7, 0, 3 \rangle$
56. $\mathbf{u} = \langle 4, 2, -3, -2, 1, 5 \rangle, \quad \mathbf{v} = \langle 3, -3, 2, -2, 1, -1 \rangle$

12.4 The Cross Product

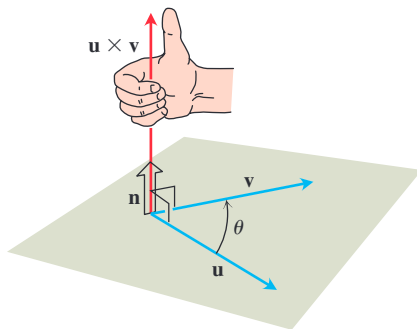


FIGURE 12.29 The construction of $\mathbf{u} \times \mathbf{v}$.

In studying lines in the plane, when we needed to describe how a line was tilting, we used the notions of slope and angle of inclination. In space, we want a way to describe how a plane is tilting. We accomplish this by multiplying two vectors in the plane together to get a third vector perpendicular to the plane. The direction of this third vector tells us the “inclination” of the plane. The product we use to multiply the vectors together is the *vector* or *cross product*, the second of the two vector multiplication methods. The cross product gives us a simple way to find a variety of geometric quantities, including volumes, areas, and perpendicular vectors. We study the cross product in this section.

The Cross Product of Two Vectors in Space

We start with two nonzero vectors \mathbf{u} and \mathbf{v} in space. Two vectors are *parallel* if one is a nonzero multiple of the other. If \mathbf{u} and \mathbf{v} are not parallel, they determine a plane. The vectors in this plane are linear combinations of \mathbf{u} and \mathbf{v} , so they can be written as a sum $a\mathbf{u} + b\mathbf{v}$. We select the unit vector \mathbf{n} perpendicular to the plane by the **right-hand rule**. This means that we choose \mathbf{n} to be the unit normal vector that points the way your right thumb points when your fingers curl through the angle θ from \mathbf{u} to \mathbf{v} (Figure 12.29). Then we define a new vector as follows.

DEFINITION The cross product $\mathbf{u} \times \mathbf{v}$ (“ \mathbf{u} cross \mathbf{v} ”) is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta) \mathbf{n}.$$

Unlike the dot product, the cross product is a vector. For this reason it is also called the **vector product** of \mathbf{u} and \mathbf{v} , and can be applied *only* to vectors in space. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u} and \mathbf{v} because it is a scalar multiple of \mathbf{n} .

There is a straightforward way to calculate the cross product of two vectors from their components. The method does not require that we know the angle between them (as suggested by the definition), but we postpone that calculation momentarily so we can focus first on the properties of the cross product.

Because the sines of 0 and π are both zero, it makes sense to define the cross product of two parallel nonzero vectors to be $\mathbf{0}$. If one or both of \mathbf{u} and \mathbf{v} are zero, we also define $\mathbf{u} \times \mathbf{v}$ to be zero. This way, the cross product of two vectors \mathbf{u} and \mathbf{v} is zero if and only if \mathbf{u} and \mathbf{v} are parallel or one or both of them are zero.

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The cross product obeys the following laws.

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r , s are scalars, then

<p>1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$</p> <p>3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$</p> <p>5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$</p>	<p>2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$</p> <p>4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$</p> <p>6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$</p>
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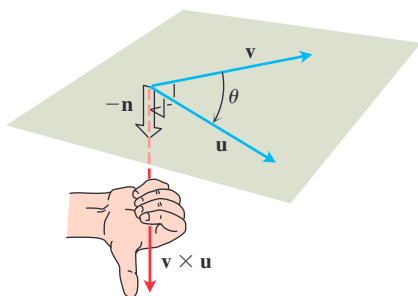


FIGURE 12.30 The construction of $\mathbf{v} \times \mathbf{u}$.

To visualize Property 3, for example, notice that when the fingers of your right hand curl through the angle θ from \mathbf{v} to \mathbf{u} , your thumb points the opposite way; the unit vector we choose in forming $\mathbf{v} \times \mathbf{u}$ is the negative of the one we choose in forming $\mathbf{u} \times \mathbf{v}$ (Figure 12.30).

Property 1 can be verified by applying the definition of cross product to both sides of the equation and comparing the results. Property 2 is proved in Appendix A.7. Property 4 follows by multiplying both sides of the equation in Property 2 by -1 and reversing the order of the products using Property 3. Property 5 is a definition. As a rule, cross product multiplication is *not associative* so $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ does not generally equal $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. (See Additional Exercise 17.)

When we apply the definition and Property 3 to calculate the pairwise cross products of \mathbf{i} , \mathbf{j} , and \mathbf{k} , we find (Figure 12.31)

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \end{aligned}$$

and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

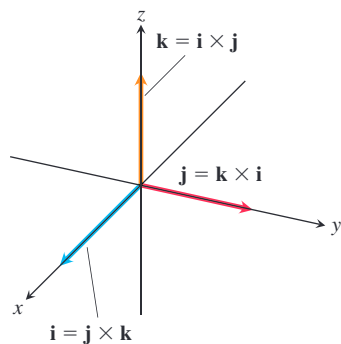


FIGURE 12.31 The pairwise cross products of \mathbf{i} , \mathbf{j} , and \mathbf{k} .

$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

Because \mathbf{n} is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin \theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin \theta.$$

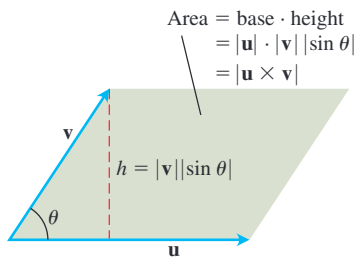


FIGURE 12.32 The parallelogram determined by \mathbf{u} and \mathbf{v} .

This is the area of the parallelogram determined by \mathbf{u} and \mathbf{v} (Figure 12.32), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}|\sin \theta$ being the height.

Determinant Formula for $\mathbf{u} \times \mathbf{v}$

Our next objective is to calculate $\mathbf{u} \times \mathbf{v}$ from the components of \mathbf{u} and \mathbf{v} relative to a Cartesian coordinate system.

Suppose that

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Then the distributive laws and the rules for multiplying \mathbf{i} , \mathbf{j} , and \mathbf{k} tell us that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} \\ &\quad + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} + u_2v_3\mathbf{j} \times \mathbf{k} \\ &\quad + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \end{aligned}$$

The component terms in the last line are hard to remember, but they are the same as the terms in the expansion of the symbolic determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

So we restate the calculation in the following easy-to-remember form.

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE 1 Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution We expand the symbolic determinant.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \\ \mathbf{v} \times \mathbf{u} &= -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \quad \text{Property 3} \end{aligned}$$

EXAMPLE 2 Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.33).

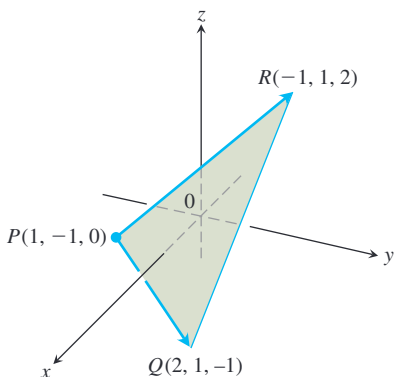


FIGURE 12.33 The vector $\overline{PQ} \times \overline{PR}$ is perpendicular to the plane of triangle PQR (Example 2). The area of triangle PQR is half of $|\overline{PQ} \times \overline{PR}|$ (Example 3).

Determinants

2×2 and 3×3 determinants are evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$

$$-a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Solution The vector $\overline{PQ} \times \overline{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\begin{aligned}\overline{PQ} &= (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\ \overline{PR} &= (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\ \overline{PQ} \times \overline{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}. \end{aligned}$$

EXAMPLE 3 Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.33).

Solution The area of the parallelogram determined by P , Q , and R is

$$\begin{aligned}|\overline{PQ} \times \overline{PR}| &= |6\mathbf{i} + 6\mathbf{k}| && \text{Values from Example 2} \\ &= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.\end{aligned}$$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE 4 Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution Since $\overline{PQ} \times \overline{PR}$ is perpendicular to the plane, its direction \mathbf{n} is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\overline{PQ} \times \overline{PR}}{|\overline{PQ} \times \overline{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

For ease in calculating the cross product using determinants, we usually write vectors in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ rather than as ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Torque

When we turn a bolt by applying a force \mathbf{F} to a wrench (Figure 12.34), we produce a torque that causes the bolt to rotate. The **torque vector** points in the direction of the axis of the bolt according to the right-hand rule (so the rotation is counterclockwise when viewed from the *tip* of the vector). The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application. The number we use to measure the torque's magnitude is the product of the length of the lever arm \mathbf{r} and the scalar component of \mathbf{F} perpendicular to \mathbf{r} . In the notation of Figure 12.34,

$$\text{Magnitude of torque vector} = |\mathbf{r}||\mathbf{F}| \sin \theta,$$

or $|\mathbf{r} \times \mathbf{F}|$. If we let \mathbf{n} be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\mathbf{r} \times \mathbf{F}$, or

$$\text{Torque vector} = \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}| \sin \theta) \mathbf{n}.$$

Recall that we defined $\mathbf{u} \times \mathbf{v}$ to be $\mathbf{0}$ when \mathbf{u} and \mathbf{v} are parallel. This is consistent with the torque interpretation as well. If the force \mathbf{F} in Figure 12.34 is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of the wrench's handle, the torque produced is zero.

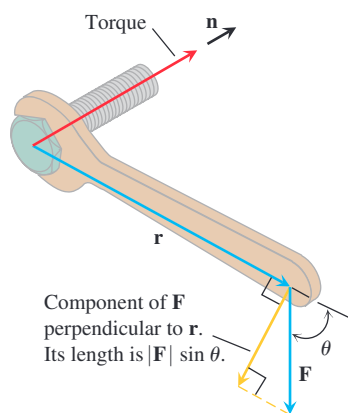


FIGURE 12.34 The torque vector describes the tendency of the force \mathbf{F} to drive the bolt forward.

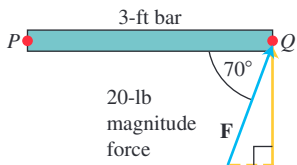


FIGURE 12.35 The magnitude of the torque exerted by \mathbf{F} at P is about 56.4 ft-lb (Example 5). The bar rotates counter-clockwise around P .

EXAMPLE 5 The magnitude of the torque generated by force \mathbf{F} at the pivot point P in Figure 12.35 is

$$|\overrightarrow{PQ} \times \mathbf{F}| = |\overrightarrow{PQ}| |\mathbf{F}| \sin 70^\circ \approx (3)(20)(0.94) \approx 56.4 \text{ ft-lb.}$$

In this example the torque vector is pointing out toward you. ■

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of \mathbf{u} , \mathbf{v} , and \mathbf{w} (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} (Figure 12.36). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base parallelogram. The number $|\mathbf{w}| |\cos \theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is also called the **box product** of \mathbf{u} , \mathbf{v} , and \mathbf{w} .

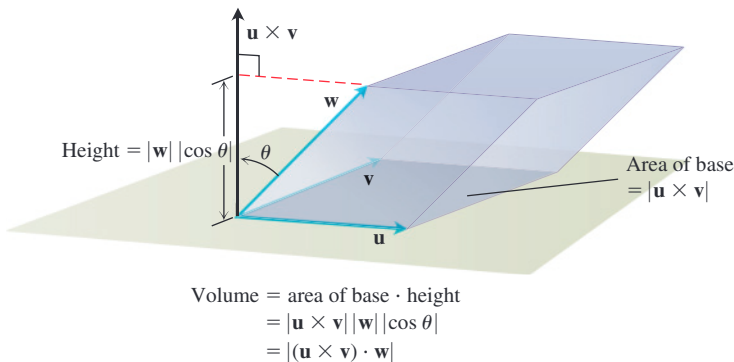


FIGURE 12.36 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

By treating the planes of \mathbf{v} and \mathbf{w} and of \mathbf{w} and \mathbf{u} as the base planes of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} , we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right) \cdot \mathbf{w} \\ &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

The dot and cross may be interchanged in a triple scalar product without altering its value.

Calculating the Triple Scalar Product as a Determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = - \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

Any two rows of a matrix can be interchanged without changing the *absolute value* of the determinant. So we can take the vectors \mathbf{u} , \mathbf{v} , \mathbf{w} in any order when calculating the absolute value of the triple product.

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) that is determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating a 3×3 determinant, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1) \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - (2) \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed. ■

EXERCISES 12.4

Cross Product Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

1. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
2. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$
3. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{0}$
5. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$
6. $\mathbf{u} = \mathbf{i} \times \mathbf{j}$, $\mathbf{v} = \mathbf{j} \times \mathbf{k}$
7. $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
8. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

In Exercises 9–14, sketch the coordinate axes and then include the vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ as vectors starting at the origin.

9. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$
10. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$
11. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
12. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
13. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
14. $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i}$

Triangles in Space

In Exercises 15–18,

- a. Find the area of the triangle determined by the points P , Q , and R .
 - b. Find a unit vector perpendicular to plane PQR .
15. $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$
 16. $P(1, 1, 1)$, $Q(2, 1, 3)$, $R(3, -1, 1)$
 17. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$
 18. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

Triple Scalar Products

In Exercises 19–22, verify that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

and find the volume of the parallelepiped (box) determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} .

\mathbf{u}	\mathbf{v}	\mathbf{w}
19. $2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
20. $\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
21. $2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
22. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

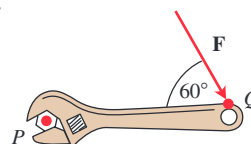
Theory and Examples

23. Parallel and perpendicular vectors Let $\mathbf{u} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{w} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.

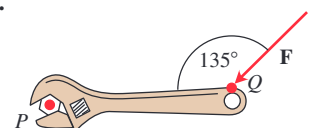
24. Parallel and perpendicular vectors Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$, $\mathbf{r} = -(\pi/2)\mathbf{i} - \pi\mathbf{j} + (\pi/2)\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.

In Exercises 25 and 26, find the magnitude of the torque exerted by \mathbf{F} on the bolt at P if $|\overline{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

25.



26.



27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

- $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$
- $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
- $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$
- $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- $(-\mathbf{u}) \times \mathbf{v} = -(\mathbf{u} \times \mathbf{v})$
- $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (any number c)
- $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (any number c)
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
- $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = 0$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

29. Given nonzero vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} , use dot product and cross product notation, as appropriate, to describe the following.

- The vector projection of \mathbf{u} onto \mathbf{v}
- A vector orthogonal to \mathbf{u} and \mathbf{v}
- A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and \mathbf{w}
- The volume of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w}
- A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$
- A vector of length $|\mathbf{u}|$ in the direction of \mathbf{v}

30. Compute $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{j} \times \mathbf{j})$. What can you conclude about the associativity of the cross product?

31. Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors. Which of the following make sense, and which do not? Give reasons for your answers.

- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
- $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

32. **Cross products of three vectors** Show that except in degenerate cases, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of \mathbf{u} and \mathbf{v} , whereas $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of \mathbf{v} and \mathbf{w} . What *are* the degenerate cases?

33. **Cancellation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

34. **Double cancellation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Area of a Parallelogram

Find the areas of the parallelograms whose vertices are given in Exercises 35–40.

- $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$
- $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$
- $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$
- $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$
- $A(0, 0, 0)$, $B(3, 2, 4)$, $C(5, 1, 4)$, $D(2, -1, 0)$
- $A(1, 0, -1)$, $B(1, 7, 2)$, $C(2, 4, -1)$, $D(0, 3, 2)$

Area of a Triangle

Find the areas of the triangles whose vertices are given in Exercises 41–47.

- $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$
- $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$
- $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$
- $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$
- $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -1)$
- $A(0, 0, 0)$, $B(-1, 1, -1)$, $C(3, 0, 3)$
- $A(1, -1, 1)$, $B(0, 1, 1)$, $C(1, 0, -1)$

48. Find the volume of a parallelepiped with one of its eight vertices at $A(0, 0, 0)$ and three adjacent vertices at $B(1, 2, 0)$, $C(0, -3, 2)$, and $D(3, -4, 5)$.

49. **Triangle area** Find a 2×2 determinant formula for the area of the triangle in the xy -plane with vertices at $(0, 0)$, (a_1, a_2) , and (b_1, b_2) . Explain your work.

50. **Triangle area** Find a concise 3×3 determinant formula that gives the area of a triangle in the xy -plane having vertices (a_1, a_2) , (b_1, b_2) , and (c_1, c_2) .

Volume of a Tetrahedron

Using the methods of Section 6.1, where volume is computed by integrating cross-sectional area, it can be shown that the volume of a tetrahedron formed by three vectors is equal to $\frac{1}{6}$ the volume of the parallelepiped formed by the three vectors. Find the volumes of the tetrahedra whose vertices are given in Exercises 51–54.

- $A(0, 0, 0)$, $B(2, 0, 0)$, $C(0, 3, 0)$, $D(0, 0, 4)$
- $A(0, 0, 0)$, $B(1, 0, 2)$, $C(0, 2, 1)$, $D(3, 4, 0)$
- $A(1, -1, 0)$, $B(0, 2, -2)$, $C(-3, 0, 3)$, $D(0, 4, 4)$
- $A(-1, 2, 3)$, $B(2, 0, 1)$, $C(1, -3, 2)$, $D(-2, 1, -1)$

In Exercises 55–57, determine whether the given points are coplanar.

- $A(1, 1, 1)$, $B(-1, 0, 4)$, $C(0, 2, 1)$, $D(2, -2, 3)$
- $A(0, 0, 4)$, $B(6, 2, 0)$, $C(2, -1, 1)$, $D(-3, -4, 3)$
- $A(0, 1, 2)$, $B(-1, 1, 0)$, $C(2, 0, -1)$, $D(1, -1, 1)$

12.5 Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space. We will use these representations throughout the rest of the text in studying the calculus of curves and surfaces in space.